IMO2024

65th International Mathematical Olympiad

SHORTLISTED PROBLEMS

WITH SOLUTIONS

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Bath, United Kingdom, 10th–22nd July 2024

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Note of Confidentiality

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Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2024 thank the following 63 countries for contributing 229 problem proposals:

Algeria, Australia, Azerbaijan, Bangladesh, Belarus, Brazil, Bulgaria, Canada, China, Colombia, Croatia, Cyprus, Czech Republic, Denmark, Dominican Republic, Ecuador, Estonia, France, Georgia, Germany, Ghana, Greece, Hong Kong, India, Indonesia, Ireland, Iran, Israel, Japan, Kazakhstan, Kosovo, Latvia, Lithuania, Luxembourg, Malaysia, Mexico, Moldova, Netherlands, New Zealand, Norway, Peru, Poland, Portugal, Romania, Senegal, Serbia, Singapore, Slovakia, Slovenia, South Africa, South Korea, Spain, Sweden, Switzerland, Syria, Taiwan, Thailand, Tunisia, Türkiye, Uganda, Ukraine, U.S.A., Uzbekistan.

Problem Selection Committee

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Problems

Algebra

A1. Determine all real numbers α such that the number

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is a multiple of n for every positive integer n. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

(Colombia)

A2. Let n be a positive integer. Find the minimum possible value of

$$S = 2^0 x_0^2 + 2^1 x_1^2 + \cdots + 2^n x_n^2,$$

where $x_0, x_1, \ldots, x_n$ are nonnegative integers such that $x_0 + x_1 + \cdots + x_n = n$.

(China)

A3. Decide whether for every sequence (a_n) of positive real numbers,

$$\frac{3^{a_1} + 3^{a_2} + \cdots + 3^{a_n}}{(2^{a_1} + 2^{a_2} + \cdots + 2^{a_n})^2} < \frac{1}{2024}$$

is true for at least one positive integer n.

(China)

A4. Let $\mathbb{Z}_{>0}$ be the set of all positive integers. Determine all subsets $\mathcal{S}$ of $\{2^0, 2^1, 2^2, \ldots\}$ for which there exists a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$\mathcal{S} = \{f(a+b) - f(a) - f(b) \mid a, b \in \mathbb{Z}_{>0}\}.$$

(Thailand)

A5. Find all periodic sequences $a_1, a_2, \ldots$ of real numbers such that the following conditions hold for all $n \geqslant 1$:

$$a_{n+2} + a_n^2 = a_n + a_{n+1}^2 \qquad \text{and} \qquad |a_{n+1} - a_n| \leqslant 1.$$

(Kosovo)

A6. Let $a_0, a_1, a_2, \ldots$ be an infinite strictly increasing sequence of positive integers such that for each $n \geqslant 1$ we have

$$a_n \in \left\{ \frac{a_{n-1} + a_{n+1}}{2}, \sqrt{a_{n-1} \cdot a_{n+1}} \right\}.$$

Let $b_1, b_2, \ldots$ be an infinite sequence of letters defined as

$$b_n = \begin{cases} A, & \text{if } a_n = \frac{1}{2}(a_{n-1} + a_{n+1}); \\ G, & \text{otherwise.} \end{cases}$$

Prove that there exist positive integers n_0 and d such that for all $n \geqslant n_0$ we have $b_{n+d} = b_n$.

(Czech Republic)

A7. Let $\mathbb{Q}$ be the set of rational numbers. Let $f\colon \mathbb{Q} \to \mathbb{Q}$ be a function such that the following property holds: for all $x, y \in \mathbb{Q}$,

$$f(x + f(y)) = f(x) + y \qquad \text{or} \qquad f(f(x) + y) = x + f(y).$$

Determine the maximum possible number of elements of $\{f(x) + f(-x) \mid x \in \mathbb{Q}\}$.

(Japan)

A8. Let $p \neq q$ be coprime positive integers. Determine all infinite sequences $a_1, a_2, \ldots$ of positive integers such that the following conditions hold for all $n \geqslant 1$:

$$\max(a_n, a_{n+1}, \ldots, a_{n+p}) - \min(a_n, a_{n+1}, \ldots, a_{n+p}) = p \qquad \text{and}$$
$$\max(a_n, a_{n+1}, \ldots, a_{n+q}) - \min(a_n, a_{n+1}, \ldots, a_{n+q}) = q.$$

(Japan)

Combinatorics

C1. Let n be a positive integer. A class of n students run n races, in each of which they are ranked with no draws. A student is eligible for a rating (a, b) for positive integers a and b if they come in the top b places in at least a of the races. Their final score is the maximum possible value of $a - b$ across all ratings for which they are eligible.

Find the maximum possible sum of all the scores of the n students.

(Australia)

C2. Let n be a positive integer. The integers $1, 2, 3, \ldots, n^2$ are to be written in the cells of an $n \times n$ board such that each integer is written in exactly one cell and each cell contains exactly one integer. For every integer d with $d \mid n$, the *d-division* of the board is the division of the board into $(n/d)^2$ nonoverlapping sub-boards, each of size $d \times d$, such that each cell is contained in exactly one $d \times d$ sub-board.

We say that n is a *cool number* if the integers can be written on the $n \times n$ board such that, for each integer d with $d \mid n$ and $1 < d < n$, in the d-division of the board, the sum of the integers written in each $d \times d$ sub-board is not a multiple of d.

Determine all even cool numbers.

(Türkiye)

C3. Let n be a positive integer. There are $2n$ knights sitting at a round table. They consist of n pairs of partners, each pair of which wishes to shake hands. A pair can shake hands only when next to each other. Every minute, one pair of adjacent knights swaps places.

Find the minimum number of exchanges of adjacent knights such that, regardless of the initial arrangement, every knight can meet her partner and shake hands at some time.

(Belarus)

C4. On a board with 2024 rows and 2023 columns, Turbo the snail tries to move from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then moves one step at a time to an adjacent cell sharing a common side. He wins if he reaches any cell in the last row. However, there are 2022 predetermined, hidden monsters in 2022 of the cells, one in each row except the first and last rows, such that no two monsters share the same column. If Turbo unfortunately reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move.

Suppose Turbo is allowed to take n attempts. Determine the minimum value of n for which he has a strategy that guarantees reaching the last row, regardless of the locations of the monsters.

(Hong Kong)

C5. Let N be a positive integer. Geoff and Ceri play a game in which they start by writing the numbers $1, 2, \ldots, N$ on a board. They then take turns to make a move, starting with Geoff. Each move consists of choosing a pair of integers (k, n), where $k \geqslant 0$ and n is one of the integers on the board, and then erasing every integer s on the board such that $2^k \mid n - s$. The game continues until the board is empty. The player who erases the last integer on the board loses.

Determine all values of N for which Geoff can ensure that he wins, no matter how Ceri plays.

(Indonesia)

C6. Let n and T be positive integers. James has $4n$ marbles with weights $1, 2, \ldots, 4n$. He places them on a balance scale, so that both sides have equal weight. Andrew may move a marble from one side of the scale to the other, so that the absolute difference in weights of the two sides remains at most T.

Find, in terms of n, the minimum positive integer T such that Andrew may make a sequence of moves such that each marble ends up on the opposite side of the scale, regardless of how James initially placed the marbles.

(Ghana)

C7. Let N be a positive integer and let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Suppose that, for each $n > N$, a_n is equal to the number of times a_{n-1} appears in the list $a_1, a_2, \ldots, a_{n-1}$.

Prove that at least one of the sequences $a_1, a_3, a_5, \ldots$ and $a_2, a_4, a_6, \ldots$ is eventually periodic.

(Australia)

C8. Let n be a positive integer. Given an $n \times n$ board, the unit cell in the top left corner is initially coloured black, and the other cells are coloured white. We then apply a series of colouring operations to the board. In each operation, we choose a 2×2 square with exactly one cell coloured black and we colour the remaining three cells of that 2×2 square black.

Determine all values of n such that we can colour the whole board black.

(Peru)

Geometry

G1. Let $ABCD$ be a cyclic quadrilateral such that $AC < BD < AD$ and $\angle DBA < 90^\circ$. Point E lies on the line through D parallel to AB such that E and C lie on opposite sides of line AD, and $AC = DE$. Point F lies on the line through A parallel to CD such that F and C lie on opposite sides of line AD, and $BD = AF$.

Prove that the perpendicular bisectors of segments BC and EF intersect on the circumcircle of $ABCD$.

(Ukraine)

G2. Let ABC be a triangle with $AB < AC < BC$, incentre I and incircle ω. Let X be the point in the interior of side BC such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point in the interior of side BC such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AB and AC, respectively.

Prove that $\angle KIL + \angle YPX = 180^\circ$.

(Poland)

G3. Let $ABCDE$ be a convex pentagon and let M be the midpoint of AB. Suppose that segment AB is tangent to the circumcircle of triangle CME at M and that D lies on the circumcircles of triangles AME and BMC. Lines AD and ME intersect at K, and lines BD and MC intersect at L. Points P and Q lie on line EC so that $\angle PDC = \angle EDQ = \angle ADB$.

Prove that lines KP, LQ, and MD are concurrent.

(Belarus)

G4. Let $ABCD$ be a quadrilateral with AB parallel to CD and $AB < CD$. Lines AD and BC intersect at a point P. Point $X \neq C$ on the circumcircle of triangle ABC is such that $PC = PX$. Point $Y \neq D$ on the circumcircle of triangle ABD is such that $PD = PY$. Lines AX and BY intersect at Q.

Prove that PQ is parallel to AB.

(Ukraine)

G5. Let ABC be a triangle with incentre I, and let Ω be the circumcircle of triangle BIC. Let K be a point in the interior of segment BC such that $\angle BAK < \angle KAC$. The angle bisector of $\angle BKA$ intersects Ω at points W and X such that A and W lie on the same side of BC, and the angle bisector of $\angle CKA$ intersects Ω at points Y and Z such that A and Y lie on the same side of BC.

Prove that $\angle WAY = \angle ZAX$.

(Uzbekistan)

G6. Let ABC be an acute triangle with $AB < AC$, and let Γ be the circumcircle of ABC. Points X and Y lie on Γ so that XY and BC intersect on the external angle bisector of $\angle BAC$. Suppose that the tangents to Γ at X and Y intersect at a point T on the same side of BC as A, and that TX and TY intersect BC at U and V, respectively. Let J be the centre of the excircle of triangle TUV opposite the vertex T.

Prove that AJ bisects $\angle BAC$.

(Poland)

G7. Let ABC be a triangle with incentre I such that $AB < AC < BC$. The second intersections of AI, BI, and CI with the circumcircle of triangle ABC are M_A, M_B, and M_C, respectively. Lines AI and BC intersect at D and lines BM_C and CM_B intersect at X. Suppose the circumcircles of triangles XM_BM_C and XBC intersect again at $S \neq X$. Lines BX and CX intersect the circumcircle of triangle SXM_A again at $P \neq X$ and $Q \neq X$, respectively.

Prove that the circumcentre of triangle SID lies on PQ.

(Thailand)

G8. Let ABC be a triangle with $AB < AC < BC$, and let D be a point in the interior of segment BC. Let E be a point on the circumcircle of triangle ABC such that A and E lie on opposite sides of line BC and $\angle BAD = \angle EAC$. Let I, I_B, I_C, J_B, and J_C be the incentres of triangles ABC, ABD, ADC, ABE, and AEC, respectively.

Prove that I_B, I_C, J_B, and J_C are concyclic if and only if AI, I_BJ_C, and J_BI_C concur.

(Canada)

Number Theory

N1. Find all positive integers n with the following property: for all positive divisors d of n, we have that $d+1 \mid n$ or $d+1$ is prime.

(Ghana)

N2. Determine all finite, nonempty sets $\mathcal{S}$ of positive integers such that for every $a, b \in \mathcal{S}$ there exists $c \in \mathcal{S}$ with $a \mid b + 2c$.

(Netherlands)

N3. Determine all sequences $a_1, a_2, \ldots$ of positive integers such that, for any pair of positive integers $m \leqslant n$, the arithmetic and geometric means

$$\frac{a_m + a_{m+1} + \cdots + a_n}{n-m+1} \quad \text{and} \quad (a_m a_{m+1} \cdots a_n)^{\frac{1}{n-m+1}}$$

are both integers.

(Singapore)

N4. Determine all positive integers a and b such that there exists a positive integer g such that $\gcd(a^n + b, b^n + a) = g$ for all sufficiently large n.

(Indonesia)

N5. Let $\mathcal{S}$ be a finite nonempty set of prime numbers. Let $1 = b_1 < b_2 < \cdots$ be the sequence of all positive integers whose prime divisors all belong to $\mathcal{S}$. Prove that, for all but finitely many positive integers n, there exist positive integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} = \left\lfloor \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \right\rfloor.$$

(Croatia)

N6. Let n be a positive integer. We say that a polynomial P with integer coefficients is *n-good* if there exists a polynomial Q of degree 2 with integer coefficients such that $Q(k)(P(k) + Q(k))$ is never divisible by n for any integer k.

Determine all integers n such that every polynomial with integer coefficients is an n-good polynomial.

(France)

N7. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Let $f \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be a function satisfying the following property: for $m, n \in \mathbb{Z}_{>0}$, the equation

$$f(mn)^2 = f(m^2) f(f(n)) f(m f(n))$$

holds if and only if m and n are coprime.

For each positive integer n, determine all the possible values of $f(n)$.

(Japan)

Solutions

Algebra

A1. Determine all real numbers α such that the number

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is a multiple of n for every positive integer n. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

(Colombia)

Answer: All even integers satisfy the condition of the problem and no other real number α does so.

Solution 1. First we will show that even integers satisfy the condition. If $\alpha = 2m$ where m is an integer then

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor = 2m + 4m + \cdots + 2mn = mn(n+1)$$

which is a multiple of n.

Now we will show that they are the only real numbers satisfying the conditions of the problem. Let $\alpha = k + \epsilon$ where k is an integer and $0 \leqslant \epsilon < 1$. Then the number

$$\begin{aligned}\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor &= k + \lfloor \epsilon \rfloor + 2k + \lfloor 2\epsilon \rfloor + \cdots + nk + \lfloor n\epsilon \rfloor \\ &= \frac{kn(n+1)}{2} + \lfloor \epsilon \rfloor + \lfloor 2\epsilon \rfloor + \cdots + \lfloor n\epsilon \rfloor\end{aligned}$$

has to be a multiple of n. We consider two cases based on the parity of k.

Case 1: k is even.

Then $\frac{kn(n+1)}{2}$ is always a multiple of n. Thus

$$\lfloor \epsilon \rfloor + \lfloor 2\epsilon \rfloor + \cdots + \lfloor n\epsilon \rfloor$$

also has to be a multiple of n.

We will prove that $\lfloor n\epsilon \rfloor = 0$ for every positive integer n by strong induction. The base case $n = 1$ follows from the fact that $0 \leqslant \epsilon < 1$. Let us suppose that $\lfloor m\epsilon \rfloor = 0$ for every $1 \leqslant m < n$. Then the number

$$\lfloor \epsilon \rfloor + \lfloor 2\epsilon \rfloor + \cdots + \lfloor n\epsilon \rfloor = \lfloor n\epsilon \rfloor$$

has to be a multiple of n. As $0 \leqslant \epsilon < 1$ then $0 \leqslant n\epsilon < n$, which means that the number $\lfloor n\epsilon \rfloor$ has to be equal to 0.

The equality $\lfloor n\epsilon \rfloor = 0$ implies $0 \leqslant \epsilon < 1/n$. Since this has to happen for all n, we conclude that $\epsilon = 0$ and then α is an even integer.

Case 2: k is odd.

We will prove that $\lfloor n\epsilon \rfloor = n-1$ for every natural number n by strong induction. The base case $n=1$ again follows from the fact that $0 \leqslant \epsilon < 1$. Let us suppose that $\lfloor m\epsilon \rfloor = m-1$ for every $1 \leqslant m < n$. We need the number

$$\begin{aligned}\frac{kn(n+1)}{2} + \lfloor \epsilon \rfloor + \lfloor 2\epsilon \rfloor + \cdots + \lfloor n\epsilon \rfloor &= \frac{kn(n+1)}{2} + 0 + 1 + \cdots + (n-2) + \lfloor n\epsilon \rfloor \\ &= \frac{kn(n+1)}{2} + \frac{(n-2)(n-1)}{2} + \lfloor n\epsilon \rfloor \\ &= \frac{k+1}{2}n^2 + \frac{k-3}{2}n + 1 + \lfloor n\epsilon \rfloor\end{aligned}$$

to be a multiple of n. As k is odd, we need $1 + \lfloor n\epsilon \rfloor$ to be a multiple of n. Again, as $0 \leqslant \epsilon < 1$ then $0 \leqslant n\epsilon < n$, so $\lfloor n\epsilon \rfloor = n-1$ as we wanted.

This implies that $1 - \frac{1}{n} \leqslant \epsilon < 1$ for all n which is absurd. So there are no other solutions in this case.

Solution 2. As in Solution 1 we check that for even integers the condition is satisfied. Then, without loss of generality we can assume $0 \leqslant \alpha < 2$. We set $S_n = \lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$.

Notice that

$$S_n \equiv 0 \pmod{n} \tag{1}$$
$$S_n \equiv S_n - S_{n-1} = \lfloor n\alpha \rfloor \pmod{n-1} \tag{2}$$

Since $\gcd(n, n-1) = 1$, (1) and (2) imply that

$$S_n \equiv n\lfloor n\alpha \rfloor \pmod{n(n-1)}. \tag{3}$$

In addition,

$$0 \leqslant n\lfloor n\alpha \rfloor - S_n = \sum_{k=1}^{n} \Big(\lfloor n\alpha \rfloor - \lfloor k\alpha \rfloor\Big) < \sum_{k=1}^{n} \Big(n\alpha - k\alpha + 1\Big) = \frac{n(n-1)}{2}\alpha + n. \tag{4}$$

For n large enough, the RHS of (4) is less than $n(n-1)$. Then (3) forces

$$0 = S_n - n\lfloor n\alpha \rfloor = \sum_{k=1}^{n} \Big(\lfloor n\alpha \rfloor - \lfloor k\alpha \rfloor\Big) \tag{5}$$

for n large enough.

Since $\lfloor n\alpha \rfloor - \lfloor k\alpha \rfloor \geqslant 0$ for $1 \leqslant k \leqslant n$, we get from (5) that, for all n large enough, all these inequalities are equalities. In particular $\lfloor \alpha \rfloor = \lfloor n\alpha \rfloor$ for all n large enough, which is absurd unless $\alpha = 0$.

Comment. An alternative ending to the previous solution is as follows.

By definition we have $S_n \leqslant \alpha\frac{n(n+1)}{2}$, on the other hand (5) implies $S_n \geqslant \alpha n^2 - n$ for all n large enough, so $\alpha = 0$.

Solution 3. As in other solutions, without loss of generality we may assume that $0 \leqslant \alpha < 2$. Even integers satisfy the condition, so we assume $0 < \alpha < 2$ and we will derive a contradiction.

By induction on n, we will simultaneously show that

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor = n^2, \tag{6}$$

$$\text{and} \qquad \frac{2n-1}{n} \leqslant \alpha < 2. \tag{7}$$

The base case is $n = 1$: If $\alpha < 1$, consider $m = \left\lceil \frac{1}{\alpha} \right\rceil > 1$, then

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor m\alpha \rfloor = 1$$

is not a multiple of m, so we deduce (7). Hence, $\lfloor \alpha \rfloor = 1$ and (6) follows.

For the induction step: assume the induction hypothesis to be true for n, then by (7)

$$2n + 1 - \frac{1}{n} \leqslant (n+1)\alpha < 2n + 2.$$

Hence,

$$n^2 + 2n \leqslant \lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor + \lfloor (n+1)\alpha \rfloor = n^2 + \lfloor (n+1)\alpha \rfloor < n^2 + 2n + 2.$$

So, necessarily $\lfloor (n+1)\alpha \rfloor = 2n + 1$ and

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor + \lfloor (n+1)\alpha \rfloor = (n+1)^2$$

in order to obtain a multiple of $n + 1$. These two equalities give (6) and (7) respectively.

Finally, we notice that condition (7) being true for all n gives a contradiction.

Solution 4. As in other solutions without loss of generality we will assume that $0 < \alpha < 2$ and derive a contradiction. For each n, we define

$$b_n = \frac{\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor}{n},$$

which is a nonnegative integer by the problem condition and our assumption. Note that

$$\lfloor (n+1)\alpha \rfloor \geqslant \lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \ldots, \lfloor n\alpha \rfloor \quad \text{and} \quad \lfloor (n+1)\alpha \rfloor > \lfloor \alpha \rfloor$$

for all $n > \frac{1}{\alpha}$. It follows that $b_{n+1} > b_n \implies b_{n+1} \geqslant b_n + 1$ for $n > \frac{1}{\alpha}$. Thus, for all such n,

$$b_n \geqslant n + C$$

where C is a fixed integer. On the other hand, the definition of b_n gives

$$b_n = \frac{\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor}{n} \leqslant \frac{\alpha + 2\alpha + \cdots + n\alpha}{n} = \frac{\alpha}{2}(n+1),$$

which is a contradiction for sufficiently large n.

A2. Let n be a positive integer. Find the minimum possible value of

$$S = 2^0x_0^2 + 2^1x_1^2 + \cdots + 2^nx_n^2,$$

where $x_0, x_1, \ldots, x_n$ are nonnegative integers such that $x_0 + x_1 + \cdots + x_n = n$.

(China)

Answer: The minimum value is $\frac{n(n+1)}{2}$.

Solution 1. For a fixed n, let $f(n)$ denote the minimum possible value of S. Consider the following variant: among all infinite sequences of nonnegative integers $x_0, x_1, \ldots$, only finitely many of which are nonzero, satisfying $x_0 + x_1 + \cdots = n$, let $g(n)$ denote the minimum possible value of

$$T = 2^0x_0^2 + 2^1x_1^2 + 2^2x_2^2 + \cdots.$$

It is clear that $g(n) \leqslant f(n)$. Conversely, it is easy to see that if a sequence $x_0, x_1, \ldots$ achieves the minimum of $g(n)$, then $x_0 \geqslant x_1 \geqslant \cdots$ and thus $x_{n+1} = x_{n+2} = \cdots = 0$. In particular, $f(n) = g(n)$.

Now, we hope to get an inductive formula for $g(n)$.

Note that, in order to minimise T for $n \geqslant 1$, we must have $x_0 \geqslant 1$ since the sequence (x_i) is nonincreasing. Note that the minimal value of

$$2^1x_1^2 + 2^2x_2^2 + \cdots = 2(2^0x_1^2 + 2^1x_2^2 + \cdots)$$

over all infinite sequences of nonnegative integers with $x_1 + x_2 + \cdots = m$ is exactly $2g(m)$. As a result, for $n \geqslant 1$ we have

$$g(n) = \min_{x_0 \in \{1,2,\ldots,n\}} \left(x_0^2 + 2g(n - x_0)\right).$$

We now prove $g(n) = \frac{n(n+1)}{2}$ by induction. It is clear that $g(0) = 0$. Assume that this has been proved for $n = 0, 1, \ldots, N-1$. Then,

$$\begin{aligned} x_0^2 + 2g(N - x_0) &= x_0^2 + (N - x_0)(N - x_0 + 1) \\ &= 2x_0^2 - (2N+1)x_0 + N(N+1) \\ &= \frac{1}{2}\left[(2x_0 - N)(2x_0 - N - 1) + N^2 + N\right]. \end{aligned} \tag{1}$$

The product of two consecutive integers $(2x_0 - N)(2x_0 - N - 1)$ is always nonnegative, and it is zero precisely when $2x_0$ is the even number in $\{N, N+1\}$. Thus the minimum of the final expression in equation (1) is $\frac{1}{2}(N^2 + N)$, so $g(N) = \frac{N(N+1)}{2}$, completing the inductive proof.

Solution 2. Consider the following table of numbers, where the row and column indices start from 0, and $a_{i,j} = 2^i(2j+1)$ for $i,\ j \geqslant 0$.

	$j=0$	1	2	3	4	5	$\cdots$
$i=0$	1	3	5	7	9	11	
1	2	6	10	14	18	22	
2	4	12	20	28	36	44	
3	8	24	40	56	72	88	
4	16	48	80	112	144	176	
$\vdots$							

Every number can be written uniquely as a product of a power of 2 and an odd number so every positive integer appears exactly once in the table above. It is easy to see that numbers in each row and each column are strictly increasing. Since the sum of the first x odd positive integers is x^2, the sum of the first x_k numbers in the k^{th} row is $2^k x_k^2$, the k^{th} term appearing in S.

Thus, the sum S can be interpreted as the result of taking a total of n numbers from the first n rows of the table such that we take the leftmost x_k numbers from row k (where $\sum_{k=1}^{n} x_k = n$), and then computing the sum of these n numbers. In particular, the minimum possible value of S is the same as the sum of the smallest n numbers in this table, since every row and every column of the table is strictly increasing.

Moreover, the smallest n numbers, namely $1, 2, \ldots, n$, appear in the first n rows, so the minimum of S is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Comment. As can be seen from the table in Solution 2, the equality case of the problem is given by

$$x_i = \left\lfloor \frac{n}{2^{i+1}} + \frac{1}{2} \right\rfloor.$$

So x_i is the result of rounding $\frac{n}{2^{i+1}}$ to the nearest integer. This also gives a proof of the identity

$$n = \sum_{i=0}^{\infty} \left\lfloor \frac{n}{2^{i+1}} + \frac{1}{2} \right\rfloor,$$

which can be separately proven by induction on n: when n is incremented by 1, exactly one term on the right hand side, namely the one corresponding to $i = \nu_2(n)$, increases by 1 while the others remain the same.

Comment. If the condition that the x_i are nonnegative integers is relaxed to the x_i being nonnegative *reals*, the problem can be solved by an application of the Cauchy-Schwarz inequality:

$$(2^0 + 2^{-1} + \cdots + 2^{-n})(2^0 x_0^2 + 2^1 x_1^2 + \cdots + 2^n x_n^2) \geqslant (x_0 + \cdots + x_n)^2 = n^2$$
$$\implies 2^0 x_0^2 + 2^1 x_1^2 + \cdots + 2^n x_n^2 \geqslant \frac{n^2}{2 - 2^{-n}}.$$

The equality case for this relaxed problem is given by

$$x_i = \frac{2^{-i} n}{2 - 2^{-n}} \approx \left\lfloor \frac{n}{2^{i+1}} + \frac{1}{2} \right\rfloor.$$

In fact, when the terms in the optimal sequence for the real case are all rounded to the nearest integer, we obtain the optimal sequence for the original problem. While thinking about the real case may guide one towards the equality case of the original problem, it does not seem like it can be easily continued into a full solution.

A3. Decide whether for every sequence (a_n) of positive real numbers,

$$\frac{3^{a_1}+3^{a_2}+\cdots+3^{a_n}}{(2^{a_1}+2^{a_2}+\cdots+2^{a_n})^2}<\frac{1}{2024} \tag{1}$$

is true for at least one positive integer n.

(China)

Comment. The question can be asked in several forms, as follows:

(i) students could be asked, as above, to show the existence of such an n;

(ii) students could be asked to show that this happens for all sufficiently large n;

(iii) students could be given a concrete positive integer N and asked to show it for all $n > N$.

The solutions below provide varying bounds for N.

Answer: The answer is "yes": there is always such an n.

Common remarks. We write $\varepsilon = \frac{1}{2024}$.

Solution 1. For every positive integer n, let $M_n = \max(a_1, a_2, \ldots, a_n)$. We first prove that

$$\frac{3^{a_1}+3^{a_2}+\cdots+3^{a_n}}{(2^{a_1}+2^{a_2}+\cdots+2^{a_n})^2} \leqslant \left(\frac{3}{4}\right)^{M_n}. \tag{2}$$

For $i = 1, 2, \ldots, n$, from $\left(\frac{3}{2}\right)^{a_i} \leqslant \left(\frac{3}{2}\right)^{M_n}$ we can obtain $3^{a_i} \leqslant \left(\frac{3}{4}\right)^{M_n} \cdot 2^{M_n} \cdot 2^{a_i}$. By summing up over all i,

$$\sum_{i=1}^{n} 3^{a_i} \leqslant \left(\frac{3}{4}\right)^{M_n} \cdot 2^{M_n} \cdot \sum_{i=1}^{n} 2^{a_i} \leqslant \left(\frac{3}{4}\right)^{M_n} \cdot \left(\sum_{i=1}^{n} 2^{a_i}\right)^2,$$

which is equivalent to (2).

Now let $\mu = \log_{4/3} \frac{1}{\varepsilon}$, so that μ is the positive real number with $\left(\frac{3}{4}\right)^{\mu} = \varepsilon$. If there is an index n such that $a_n > \mu$, then $M_n \geqslant a_n > \mu$, and hence

$$\frac{3^{a_1}+3^{a_2}+\cdots+3^{a_n}}{(2^{a_1}+2^{a_2}+\cdots+2^{a_n})^2} \leqslant \left(\frac{3}{4}\right)^{M_n} < \left(\frac{3}{4}\right)^{\mu} = \varepsilon.$$

Otherwise we have $0 < a_i \leqslant \mu$ for all positive integers i, so

$$\frac{3^{a_1}+3^{a_2}+\cdots+3^{a_n}}{(2^{a_1}+2^{a_2}+\cdots+2^{a_n})^2} \leqslant \frac{n \cdot 3^{\mu}}{(n \cdot 1)^2} = \frac{3^{\mu}}{n}.$$

If $n > \left\lfloor \frac{3^{\mu}}{\varepsilon} \right\rfloor$, this is less than ε.

Comment. It is also possible to prove (2) by induction on n. The base case $n = 1$ is clear. For the induction step, after ordering $a_1, a_2, \ldots, a_n$ in increasing order as $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$, it suffices, for example, to verify that

$$\frac{3^{b_1}+3^{b_2}+\cdots+3^{b_n}}{(2^{b_1}+2^{b_2}+\cdots+2^{b_n})^2} \leqslant \frac{3^{b_1}+3^{b_2}+\cdots+3^{b_n}}{(2^{b_1}+2^{b_2}+\cdots+2^{b_n})(2^{b_2}+\cdots+2^{b_n})} \leqslant \frac{3^{b_2}+\cdots+3^{b_n}}{(2^{b_2}+\cdots+2^{b_n})^2}.$$

The second inequality is equivalent to $3^{b_1} \sum_{i=2}^{n} 2^{b_i} \leqslant 2^{b_1} \sum_{i=2}^{n} 3^{b_i}$, which follows from $\left(\frac{3}{2}\right)^{b_1} \leqslant \left(\frac{3}{2}\right)^{b_i}$.

Solution 2. We will combine two upper bounds.

First, start with the trivial estimate

$$\frac{3^{a_1}+\cdots+3^{a_n}}{(2^{a_1}+\cdots+2^{a_n})^2} \leqslant \frac{3^{a_1}+\cdots+3^{a_n}}{4^{a_1}+\cdots+4^{a_n}}.$$

By applying Jensen's inequality to the convex function $x^{\log_3 4}$ we get

$$\frac{4^{a_1}+\cdots+4^{a_n}}{n} = \frac{\left(3^{a_1}\right)^{\log_3 4}+\cdots+\left(3^{a_n}\right)^{\log_3 4}}{n} \geqslant \left(\frac{3^{a_1}+\cdots+3^{a_n}}{n}\right)^{\log_3 4},$$

so

$$\frac{3^{a_1}+\cdots+3^{a_n}}{(2^{a_1}+\cdots+2^{a_n})^2} \leqslant \frac{3^{a_1}+\cdots+3^{a_n}}{4^{a_1}+\cdots+4^{a_n}} \leqslant \left(\frac{n}{3^{a_1}+\cdots+3^{a_n}}\right)^{\log_3 4-1}.$$

Hence, (1) holds true whenever

$$3^{a_1}+\cdots+3^{a_n} > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\log_3 4-1}} \cdot n. \tag{3}$$

Second, trivially

$$\frac{3^{a_1}+\cdots+3^{a_n}}{(2^{a_1}+\cdots+2^{a_n})^2} \leqslant \frac{3^{a_1}+\cdots+3^{a_n}}{n^2},$$

so (1) is satisfied if

$$3^{a_1}+\cdots+3^{a_n} < \varepsilon \cdot n^2. \tag{4}$$

If $n > \left(\frac{1}{\varepsilon}\right)^{1+\frac{1}{\log_3 4-1}}$ then $\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\log_3 4-1}} \cdot n < \varepsilon \cdot n^2$, and therefore at least one of (3) and (4) is satisfied.

Solution 3. Define $C = \log_{4/3} \frac{2}{\varepsilon}$, so that if $a_i > C$ then $3^{a_i} < \frac{\varepsilon}{2} \cdot 4^{a_i}$. We divide the sequence into "small" and "large" terms by how they compare to C: let

$$\mathcal{S}_n = \{i \leqslant n \mid a_i \leqslant C\} \quad \text{and} \quad \mathcal{L}_n = \{i \leqslant n \mid a_i > C\}.$$

Then (1) is equivalent to

$$\frac{\sum_{i\in\mathcal{S}_n} 3^{a_i}}{\left(\sum_{i\in\mathcal{S}_n} 2^{a_i}+\sum_{i\in\mathcal{L}_n} 2^{a_i}\right)^2} + \frac{\sum_{i\in\mathcal{L}_n} 3^{a_i}}{\left(\sum_{i\in\mathcal{S}_n} 2^{a_i}+\sum_{i\in\mathcal{L}_n} 2^{a_i}\right)^2} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}.$$

If $\mathcal{L}_n$ is nonempty, we have

$$\frac{\sum_{i\in\mathcal{L}_n} 3^{a_i}}{\left(\sum_{i\in\mathcal{S}_n} 2^{a_i}+\sum_{i\in\mathcal{L}_n} 2^{a_i}\right)^2} < \frac{\varepsilon}{2} \cdot \frac{\sum_{i\in\mathcal{L}_n} 4^{a_i}}{\left(\sum_{i\in\mathcal{L}_n} 2^{a_i}\right)^2} \leqslant \frac{\varepsilon}{2},$$

and this also trivially holds when $\mathcal{L}_n$ is empty (in which case the LHS is zero).

Now suppose that $n \geqslant \frac{2}{\varepsilon}\left(\frac{3}{2}\right)^C$. Note that $3^{a_i} \leqslant \left(\frac{3}{2}\right)^C 2^{a_i}$ for $i \in \mathcal{S}_n$, so we have

$$\frac{\sum_{i\in\mathcal{S}_n} 3^{a_i}}{\left(\sum_{i\in\mathcal{S}_n} 2^{a_i}+\sum_{i\in\mathcal{L}_n} 2^{a_i}\right)^2} \leqslant \frac{\left(\frac{3}{2}\right)^C \sum_{i\in\mathcal{S}_n} 2^{a_i}}{\left(\sum_{i\in\mathcal{S}_n} 2^{a_i}+\sum_{i\in\mathcal{L}_n} 2^{a_i}\right)^2} \leqslant \frac{\left(\frac{3}{2}\right)^C}{\sum_{i\in\mathcal{S}_n} 2^{a_i}+\sum_{i\in\mathcal{L}_n} 2^{a_i}} < \frac{\left(\frac{3}{2}\right)^C}{n} \leqslant \frac{\varepsilon}{2},$$

so we have (1).

Solution 4. For every index $i = 1, 2, \ldots, n$, apply the weighted AM-GM inequality to numbers 2^{a_i} and $(n-1)$ with weights $\log_2 \frac{3}{2} \approx 0.585$ and $\log_2 \frac{4}{3} \approx 0.415$ as

$$\begin{aligned}2^{a_1} + 2^{a_2} + \cdots + 2^{a_n} &\geqslant 2^{a_i} + (n-1)\\ > \log_2 \frac{3}{2} \cdot 2^{a_i} + \log_2 \frac{4}{3} \cdot (n-1) &\geqslant \left(2^{a_i}\right)^{\log_2 \frac{3}{2}} \cdot (n-1)^{\log_2 \frac{4}{3}}\\ = \left(\frac{3}{2}\right)^{a_i} \cdot (n-1)^{\log_2 \frac{4}{3}} &> \left(\frac{3}{2}\right)^{a_i} \cdot (n-1)^{2/5}.\end{aligned}$$

By summing up for $i = 1, 2, \ldots, n$,

$$(2^{a_1} + \cdots + 2^{a_n})^2 = \sum_{i=1}^{n} 2^{a_i}\left(2^{a_1} + 2^{a_2} + \cdots + 2^{a_n}\right) > (n-1)^{2/5} \sum_{i=1}^{n} 3^{a_i}$$

so

$$\frac{3^{a_1} + 3^{a_2} + \cdots + 3^{a_n}}{(2^{a_1} + 2^{a_2} + \cdots + 2^{a_n})^2} < \frac{1}{(n-1)^{2/5}}.$$

If $n \geqslant \left(\frac{1}{\varepsilon}\right)^{5/2} + 1$ then $\frac{1}{(n-1)^{2/5}} < \varepsilon$.

A4. Let $\mathbb{Z}_{>0}$ be the set of all positive integers. Determine all subsets $\mathcal{S}$ of $\{2^0, 2^1, 2^2, \dots\}$ for which there exists a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$\mathcal{S} = \{f(a+b) - f(a) - f(b) \mid a, b \in \mathbb{Z}_{>0}\}.$$

(Thailand)

Answer: $\mathcal{S}$ can be any subset of size 1 or 2.

Common remarks. For this problem, it is convenient to use notation such as $\{a, b, c\}$ for multisets rather than sets, and the subset relation is likewise that for multisets. Both solutions use the following property of powers of 2: if $2^a + 2^b = 2^c + 2^d$, then $\{a, b\}$ and $\{c, d\}$ are the same multiset. Define $e(a, b) = \log_2(f(a+b) - f(a) - f(b)) = e(b, a)$. Thus,

$$f(a+b) = 2^{e(a,b)} + f(a) + f(b).$$

Solution 1. Clearly $\mathcal{S}$ must be nonempty. We start with constructions when $1 \leqslant |\mathcal{S}| \leqslant 2$.

- If $\mathcal{S} = \{2^k\}$, then take $f(x) = cx - 2^k$ for any integer $c > 2^k$.
- If $\mathcal{S} = \{2^k, 2^\ell\}$ where $k > \ell$, then take $f(x) = (2^k - 2^\ell)\lfloor \alpha x \rfloor - 2^\ell$, where $\alpha > 2$ is not an integer. This works because $\lfloor \alpha(x+y) \rfloor - (\lfloor \alpha x \rfloor + \lfloor \alpha y \rfloor) \in \{0, 1\}$ for all x and y, and takes both values; the lower bound on α ensures the values of f are positive.

Observe that, inductively,

$$f(n) = 2^{e(1,1)} + 2^{e(2,1)} + \cdots + 2^{e(n-1,1)} + nf(1).$$

Lemma 1. For any positive integers n and k,

$$\{e(1,1), e(2,1), \dots, e(k-1,1)\} \subset \{e(n,1), e(n+1,1), \dots, e(n+k-1,1)\}.$$

Proof. We work by induction on k; in the case $k = 1$, the first multiset is empty, which provides our base case.

For the induction step, suppose $k \geqslant 2$ and we know that

$$\{e(1,1), e(2,1), \dots, e(k-2,1)\} \subset \{e(n,1), e(n+1,1), \dots, e(n+k-2,1)\}.$$

By definition, $f(n+k) - f(n) - f(k) = 2^{e(n,k)}$, and using the first observation we see that

$$f(n+k) - f(n) - f(k) = \left(2^{e(n,1)} + 2^{e(n+1,1)} + \cdots 2^{e(n+k-1,1)}\right) - \left(2^{e(1,1)} + 2^{e(2,1)} + \cdots + 2^{e(k-1,1)}\right).$$

From the induction hypothesis, we may write

$$\{e(n,1), e(n+1,1), \dots, e(n+k-2,1)\} = \{e(1,1), e(2,1), \dots, e(k-2,1)\} \cup \{a\}$$

for some a. Thus

$$2^{e(n,k)} = 2^a + 2^{e(n+k-1,1)} - 2^{e(k-1,1)}.$$

So $\{e(n,k), e(k-1,1)\} = \{a, e(n+k-1,1)\}$. Thus $e(k-1,1) = a$ or $e(k-1,1) = e(n+k-1,1)$, and in either case we have our result. □

Lemma 2. The sequence $e(1,1)$, $e(2,1)$, $e(3,1)$, … takes at most two different values.

Proof. Suppose for a contradiction that $k \geqslant 2$ is the least index with $e(k,1) \neq e(1,1)$, and that some $\ell > k$ has $e(\ell,1) \notin \{e(k,1), e(1,1)\}$. By Lemma 1, any block of k consecutive values of the sequence has at least $k-1$ values equal to $e(1,1)$. This forces

$$e(\ell-1,1) = e(\ell-2,1) = \cdots = e(\ell-(k-1),1) = e(1,1)$$

and

$$e(\ell+1,1) = e(\ell+2,1) = \cdots = e(\ell+(k-1),1) = e(1,1).$$

But then the block $e(\ell-1,1)$, $e(\ell,1)$, $e(\ell+1,1)$, $e(\ell+2,1)$, …, $e(\ell+(k-1),1)$ has length $k+1$ and does not contain $e(k,1)$, a contradiction. □

Finally, for any a and b we have

$$\begin{aligned} f(a+b) - f(a) - f(b) &= \left(2^{e(a,1)} + 2^{e(a+1,1)} + \cdots 2^{e(a+b-1,1)}\right) - \left(2^{e(1,1)} + 2^{e(2,1)} + \cdots + 2^{e(b-1,1)}\right) \\ &= 2^{e(i,1)} \end{aligned}$$

for some $a \leqslant i \leqslant a+b-1$. So $|\mathcal{S}| \leqslant 2$.

Comment. In the construction of functions, $\alpha > 2$ is only necessary if $k = \ell+1$, to make sure $f(1) \neq 0$. Otherwise, any nonintegral $\alpha > 1$ suffices.

Solution 2. Subsets of size 1 or 2 can be achieved as in Solution 1, and $\mathcal{S}$ must be nonempty. We consider such a set $\mathcal{S}$ with $|\mathcal{S}| \geqslant 3$ and a corresponding function f in order to achieve a contradiction. We will relate the $e(a,b)$ to values of $e(c,1)$ with $c+1 < a+b$, leading to a proof of Lemma 2 from Solution 1 that does not depend on Lemma 1 from that solution.

Suppose $a > 1$. We have $f(a+b) - f(a) - f(b) = 2^{e(a,b)}$ and also $f(a) - f(a-1) - f(1) = 2^{e(a-1,1)}$, so

$$f(a+b) - f(a-1) - f(1) - f(b) = 2^{e(a,b)} + 2^{e(a-1,1)}.$$

Similarly, $f(a+b) - f(a-1) - f(b+1) = 2^{e(a-1,b+1)}$ and $f(b+1) - f(1) - f(b) = 2^{e(b,1)}$, so

$$f(a+b) - f(a-1) - f(1) - f(b) = 2^{e(a-1,b+1)} + 2^{e(b,1)}.$$

Thus either

$$e(a,b) = e(a-1,b+1) \quad \text{and} \quad e(a-1,1) = e(b,1)$$

or

$$e(a,b) = e(b,1) \quad \text{and} \quad e(a-1,b+1) = e(a-1,1).$$

For $n \geqslant 4$, we consider these possibilities as (a,b) ranges over all pairs with $a+b=n$. If the first case holds for every such pair (that is, if $e(c,1) = e(d,1)$ for all $c+d = n-1$), then all the $e(a,b)$ for $a+b=n$ are equal (and the above equations do not constrain whether or not the value is the same as any $e(c,1)$ with $c+1 < n$). Otherwise, the values of $e(a,b)$ with $a+b=n$ are fully determined by the values of $e(c,1)$ for which $e(c,1) \neq e(n-1-c,1)$, and are not all equal.

Specifically, if $e(c,1) = j$ and $e(n-1-c,1) = k$ with $j \neq k$, we have $e(c,n-c) = j = e(n-c,c)$ and $e(c+1,n-c-1) = k = e(n-c-1,c+1)$. Every other value of $e(a,b)$ with $a+b=n$ is then determined by the rule that $e(a,b) = e(a-1,b+1)$ if $e(a-1,1) = e(b,1)$: if we have $e(c,1) \neq e(n-1-c,1)$, and $e(c',1) \neq e(n-1-c',1)$, but $e(d,1) = e(n-1-d,1)$ for all $c < d < c'$, then if $c < c'-1$ we have $e(c'-1, n-(c'-1)) = e(c', n-c')$, then if $c < c'-2$ we have $e(c'-2, n-(c'-2)) = e(c'-1, n-(c'-1)) = e(c', n-c')$, and so on until $e(c+1, n-c-1) = e(c', n-c')$ (yielding a contradiction if $e(n-c-1,1) \neq e(c',1)$; such a contradiction also arises trivially if $c+1 = c'$ and $e(n-c-1,1) \neq e(c',1)$). If c is the least integer such that $e(c,1) \neq e(n-1-c,1)$, the values of $e(a,b)$ with $a < c$ are similarly determined to be equal to $e(c,n-c)$ (and likewise for $a > n-c$).

In other words, if we list the values in ascending order of a from 1 to $n-1$, any gaps between the pairs of adjacent values determined when $e(c,1) \neq e(n-1-c,1)$ are filled with copies of the previously determined adjacent values, and if the values on either side of such a gap are different, we have a contradiction (including in the degenerate cases where the pairs are adjacent or overlap, if $c+1=c'$). Note in particular that every value of $e(a,b)$ is a value of $e(c,1)$ for some c with $c+1 \leqslant a+b$.

If $|\mathcal{S}| \geqslant 3$, that means that $e(c,1)$ takes at least three different values. Let m be such that $e(m,1)$ does not equal any $e(c,1)$ for $c<m$, and there are exactly two different values of $e(c,1)$ for $c<m$ (and thus $m \geqslant 3$).

Because $e(m,1)$ does not equal any $e(c,1)$ for $c<m$, we have that all $e(a,b)$ for $a+b=m+1$ are equal, and $e(c,1)=e(d,1)$ for all $c+d=m$. We now consider the values of $e(a,b)$ for $a+b=m+2$ determined by the above rules. Since $e(m,1) \neq e(1,1)$, we have $e(1,m+1)=e(1,1)$ and $e(2,m)=e(m,1)$. If there were any other $e(d,1) \neq e(m+1-d,1)$, consider the one with minimal $d>1$; because $e(m,1) \neq e(d,1)$, we arrive at a contradiction. So every $e(c,1)=e(d,1)$ for $c+d=m+1$ except for $e(m,1) \neq e(1,1)$. But these equalities form a path connecting all $e(c,1)$ for $c<m$:

$$e(1,1)=e(m-1,1)=e(2,1)=e(m-2,1)=e(3,1)=\cdots$$

which contradicts the assumption we made that there were exactly two different values of $e(c,1)$ for $c<m$.

Solution 3. Constructions for $1 \leqslant |\mathcal{S}| \leqslant 2$ are shown in Solution 1, and $\mathcal{S}$ must be nonempty. We suppose $|\mathcal{S}| \geqslant 3$ to derive a contradiction.

Claim 1. $e(a,b)$, $e(b,c)$, and $e(a,c)$ can take at most two different values.

Proof. By expanding $f(a+b+c)$ in three different ways, we get

$$2^{e(a,b)}+2^{e(c,a+b)}=2^{e(b,c)}+2^{e(a,b+c)}=2^{e(a,c)}+2^{e(b,a+c)}.$$

The result follows from the equality of the three multisets of exponents. □

For Claims 2 to 4, we fix k and let N be the smallest integer such that $e(a,N-a+1)=k$ for some $a \leqslant N$.

Claim 2. For any b with $b \leqslant N$, we must have $e(b,N-b+1)=k$.

Proof. Suppose that $e(a,N-a+1)=k$ and $a<b$. Expanding $f(a+(b-a)+N-b+1)$ in two different ways, we see that

$$2^{e(a,b-a)}+2^{e(b,N-b+1)}=2^{e(N-b+1,b-a)}+2^{e(N-a+1,a)}.$$

By the minimality of N, we must have $e(b,N-b+1)=e(N-a+1,a)$. The case of $a>b$ follows by replacing a and b with $N-a+1$ and $N-b+1$. □

Claim 3. $e(a,1)=e(N-a+1,1)$ for any a satisfying $1<a<N$.

Proof. By Claim 2, $e(a,N-a+1)=k$. Then by Claim 1, $e(a,N-a+1)$, $e(a,1)$, and $e(N-a+1,1)$ can take at most two different values. But by the minimality of N, we must have $e(a,N-a+1) \neq e(a,1)=e(N-a+1,1)$. □

Claim 4. $e(a,1)=e(N-a,1)$ for any a satisfying $1 \leqslant a<N$.

Proof. By Claim 2, $e(a,N-a+1)=e(a+1,N-a)=k$. Expanding $f(1+a+(N-a))$ in two different ways, we see that

$$2^{e(1,a)}+2^{e(a+1,N-a)}=2^{e(1,N-a)}+2^{e(N-a+1,a)}.$$

Therefore $e(1,a)=e(1,N-a)$, as required. □

If $|\mathcal{S}| \geqslant 3$, then there exist $1<N_k<N_\ell$ where N_k and N_ℓ are the minimal values corresponding to k and ℓ. But Claims 3 and 4 imply that $e(a,1)$ is constant for all $1 \leqslant a<N_\ell$, which is a contradiction.

A5. Find all periodic sequences $a_1,\ a_2,\ \ldots$ of real numbers such that the following conditions hold for all $n \geqslant 1$:

$$a_{n+2} + a_n^2 = a_n + a_{n+1}^2 \qquad \text{and} \qquad |a_{n+1} - a_n| \leqslant 1.$$

(Kosovo)

Answer: The sequences satisfying the conditions of the problem are:

$$\begin{gathered} c, -c, c, -c, \ldots, \\ d, d, d, d, \ldots, \end{gathered}$$

where $c \in [-\frac{1}{2}, \frac{1}{2}]$ and d is any real number.

Solution 1. We rewrite the first condition as

$$a_{n+2} + a_{n+1} = (a_{n+1} + a_n)(a_{n+1} - a_n + 1). \tag{1}$$

If there exists a positive integer m such that $a_{m+1} + a_m = 0$, then from equation (1) we have $a_{n+1} + a_n = 0$ for all positive integers $n \geqslant m$. By the fact that the sequence $(a_{i+1} + a_i)$ is periodic, we get $a_{i+1} + a_i = 0$ for every positive integer i. Thus the sequence (a_i) is of the form $c,\ -c,\ c,\ -c,\ \ldots$ for some $|c| \leqslant \frac{1}{2}$.

Now suppose that $a_{n+1} + a_n \neq 0$ for every positive integer n. Let T be the period of the sequence. From equation (1) we have

$$1 = \prod_{i=1}^{T} \frac{a_{i+2} + a_{i+1}}{a_{i+1} + a_i} = \prod_{i=1}^{T} (a_{i+1} - a_i + 1).$$

Combining with the second condition $|a_{i+1} - a_i| \leqslant 1$, we have $a_{i+1} - a_i + 1 > 0$. Using the AM-GM inequality we get

$$1 = \prod_{i=1}^{T} (a_{i+1} - a_i + 1) \leqslant \left(\frac{\sum_{i=1}^{T} (a_{i+1} - a_i + 1)}{T} \right)^T = 1.$$

So the equality holds, and thus we get

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_{T+1} - a_T,$$

which means that (a_i) is a constant sequence.

So all sequences satisfying the conditions of the problem are those listed above.

Solution 2. Define $s_n = a_{n+1} + a_n$ and $d_n = a_{n+1} - a_n$, so the original sequence is periodic if and only if both these sequences are periodic. Rearranging the given conditions yields $s_{n+1} = s_n(1 + d_n)$ and $d_{n+1} = d_n(s_n - 1)$, with $|d_n| \leqslant 1$ for all n.

If $s_n = 0$ for some n then $s_i = 0$ for all $i \geqslant n$, and for the sequence to be periodic we must have all $s_i = 0$ and the sequence $c, -c, c, -c, \ldots$, for some $|c| \leqslant \frac{1}{2}$. Similarly, if $d_n = 0$ for some n and the sequence is periodic, then all $d_i = 0$ and the sequence is $c, c, c, \ldots$.

We claim those are the only periodic sequences, so suppose for contradiction that we have a periodic sequence where no s_i or d_i is 0. Under this hypothesis, we will prove that (s_n), (d_n) have the following properties.

1. All s_n are positive numbers. As $|d_n| \leqslant 1$ and $s_{n+1} = s_n(1+d_n) \neq 0$, it follows that $d_n > -1$ and that all s_n have the same sign (all positive or all negative). If all s_n are negative, then $|d_{n+1}| = |d_n(s_n - 1)| > |d_n|$, so $|d_i|$ is a strictly increasing sequence, contradicting periodicity.

2. Whenever $d_n > 0$ we have $0 < s_n < 1$. Suppose for contradiction that we have $d_n > 0$ and $s_n \geqslant 1$ for some n. Since $d_{n+1} \neq 0$ we have $s_n > 1$, and then $d_{n+1} > 0$, $s_{n+1} > s_n > 1$. Inductively, all $d_i > 0$ for $i \geqslant n$, and $s_{i+1} > s_i$ for $i \geqslant n$, contradicting periodicity.

Now we can get the desired contraction as follows. Suppose that the period of (a_i) is T, then $\sum_{i=1}^{T} d_i = a_{T+1} - a_1 = 0$, hence there is an n such that $d_n > 0$. By property 2 we get $0 < s_n < 1$, and in particular $s_n < 2$. Suppose that we have $s_i < 2$. If $d_i < 0$, then $s_{i+1} = s_i(1+d_i) < s_i < 2$; if $d_i > 0$, then by property 2 we have $0 < s_i < 1$, and then $s_{i+1} = s_i(1 + d_i) \leqslant 2s_i < 2$. In both cases we get $s_{i+1} < 2$, and then by induction we get $s_k < 2$ for all $k \geqslant n$. But then we have $|d_{k+1}| = |d_k(s_k - 1)| < |d_k|$, which contradicts the fact that (d_k) is periodic.

So the only periodic sequences are the two listed above.

Solution 3. Note that if $a_{n+1} = -a_n$ for any n, then $a_{n+2} = a_n = -a_{n+1}$, yielding the first answer by periodicity. Also, if $a_{n+1} = a_n$ for any n, then $a_{n+2} = a_n = a_{n+1}$, yielding the second answer by periodicity. If $a_{n+2} = a_n$ for any n, then $a_n^2 = a_{n+1}^2$ so $a_{n+1} = \pm a_n$ and one of those two cases applies. Henceforth, we will assume that the sequence is neither one of the answers and $a_n \neq a_{n+1}, -a_{n+1}, a_{n+2}$ for all n for the rest of the solution. Note that the recursion rearranges to

$$a_{n+2} - a_{n+1} = (a_{n+1}^2 - a_{n+1}) - (a_n^2 - a_n) = (a_{n+1} - a_n)(a_{n+1} + a_n - 1). \tag{2}$$

Claim 1. We have that $a_n \leqslant \frac{1}{2}$ for all n.

Proof. First, we cannot have $a_n > \frac{1}{2}$ for all n. Otherwise, $a_{n+1} + a_n - 1$ is positive for all n, so (2) implies that $a_{n+2} - a_{n+1}$ has the same sign as $a_{n+1} - a_n$ for all n. This would mean that the sequence is monotonic, contradicting periodicity.

On the other hand, if $a_{n+1} \leqslant \frac{1}{2}$ and $a_{n+2} > \frac{1}{2}$, then

$$a_{n+1}^2 = a_{n+2} + a_n^2 - a_n \geqslant a_{n+2} - \frac{1}{4} > \frac{1}{4} \implies |a_{n+1}| > \frac{1}{2},$$

where we use the fact that $t^2 - t \geqslant -\frac{1}{4}$ for all $t \in \mathbb{R}$. As $a_{n+1} \leqslant \frac{1}{2}$, this means that $a_{n+1} < -\frac{1}{2}$ so $|a_{n+1} - a_{n+2}| > 1$, a contradiction. □

The identity (2) now implies that $a_{n+2} - a_{n+1}$ and $a_{n+1} - a_n$ are of opposite signs for all n, so that $a_n < a_{n+1} \iff a_{n+1} > a_{n+2}$.

Claim 2. We have that $a_n > 0 \iff a_{n+1} \leqslant 0$: that is, the signs of the sequence are alternating.

Proof. First, it cannot be the case that $a_n > 0$ for all n. Indeed, then we would have from Claim 1 that $|a_{n+1} + a_n - 1| < 1$ for all n, which by (2) means that $|a_{n+1} - a_n|$ is strictly decreasing in n, a contradiction of the sequence's periodicity. It also cannot be the case that $a_n \leqslant 0$ for all n, as then we would have that $|a_{n+1} + a_n - 1| > 1$ for all n (noting that by the nonconstant assumption we will never have $a_n = a_{n+1} = 0$) so $|a_{n+1} - a_n|$ is strictly increasing in n.

Hence, if the signs of a_n are not alternating, then by periodicity there exists n with $a_n > 0$ and $a_{n+1}, a_{n+2} \leqslant 0$ or $a_n, a_{n+1} > 0$ and $a_{n+2} \leqslant 0$. In either scenario, we have that

$$a_n^2 - a_n = a_{n+1}^2 - a_{n+2} \geqslant a_{n+1}^2 \geqslant 0 \implies a_n > 1$$

as a_n is positive and a_{n+2} is nonpositive.

In the former case, we have that $a_n - a_{n+1} > 1$, a contradiction. In the latter case, as $a_{n+1} > a_{n+2}$, we must have that $a_n < a_{n+1}$. But then we have that $a_{n+1} - a_{n+2} > a_n - a_{n+2} > 1$, a contradiction. □

Note now that we cannot have $a_{n+2} > -a_{n+1} > a_n$ for any n, as we would then have

$$a_{n+1}^2 - a_n^2 = a_{n+2} - a_n > -a_{n+1} - a_n > 0 \implies a_n - a_{n+1} > 1,$$

a contradiction. Similarly, we cannot have $a_n > -a_{n+1} > a_{n+2}$ for any n, as we would then have

$$a_n^2 - a_{n+1}^2 = a_n - a_{n+2} > a_n + a_{n+1} > 0 \implies a_n - a_{n+1} > 1.$$

Having ruled out these scenarios, we may conclude that $|a_{n+1}|$ is not between $|a_n|$ and $|a_{n+2}|$ for any n.

Let k be an index such that $|a_k|$ is maximal. Note that we cannot have $|a_{k-2}| = |a_k|$, as that would imply that $a_{k-2} = a_k$ by Claim 2. We also cannot have $|a_{k-2}| \leqslant |a_{k-1}|$, as that would imply that $|a_{k-1}|$ is between $|a_{k-2}|$ and $|a_k|$. Hence, we must have that $|a_{k-1}| < |a_{k-2}| < |a_k|$. As $|a_k|$ is maximal, we cannot have $a_k = 0$. If $a_k > 0$, then we have that $a_k - a_{k-2} = a_{k-1}^2 - a_{k-2}^2 < 0$, a contradiction. If $a_k < 0$, then we have that $a_k - a_{k+2} = a_k^2 - a_{k+1}^2 > 0$, a contradiction.

A6. Let $a_0, a_1, a_2, \ldots$ be an infinite strictly increasing sequence of positive integers such that for each $n \geqslant 1$ we have

$$a_n \in \left\{ \frac{a_{n-1}+a_{n+1}}{2}, \sqrt{a_{n-1} \cdot a_{n+1}} \right\}.$$

Let $b_1, b_2, \ldots$ be an infinite sequence of letters defined as

$$b_n = \begin{cases} A, & \text{if } a_n = \frac{1}{2}(a_{n-1}+a_{n+1}); \\ G, & \text{otherwise.} \end{cases}$$

Prove that there exist positive integers n_0 and d such that for all $n \geqslant n_0$ we have $b_{n+d} = b_n$.

(Czech Republic)

Common remarks. In fact, all known proofs proceed by showing that the eventual period of the sequence (b_n) always consists of some number of occurrences of G (possibly zero) followed by an A.

Such sequences of any period $p \geqslant 1$ exist. Indeed, consider the sequence

$$\ldots, k^p, k^{p-1}(k+1), \ldots, k(k+1)^{p-1}, (k+1)^p, (k+1)^{p-1}(k+2), \ldots.$$

The Tournament of the Towns, in Spring 2009 (Junior A-Level Paper, problem 4), considered sequences satisfying exactly this fairly natural criterion. However, it asked a vastly easier question about them: in the language of this problem, it asked whether every such sequence had (b_n) eventually constant. The answer to that problem is "no", as heavily hinted by the statement of this problem. Thus, at least so far as the Problem Selection Committee knows, this is a novel problem about a family of sequences which has been previously considered.

Solution 1. We will show that the eventual period of sequence (b_n) consists of any fixed number of occurrences of G (possibly zero) followed by a single A.

We look at the ratios of consecutive terms of the sequence (a_n). Let C and D be coprime positive integers such that $a_1/a_0 = (C+D)/C$. If $b_n = G$ then $a_n/a_{n-1} = a_{n+1}/a_n$. If $b_n = A$ and $a_n/a_{n-1} = (C+kD)/(C+(k-1)D)$ for some positive integer k then

$$\frac{a_{n+1}}{a_n} = \frac{2a_n - a_{n-1}}{a_n} = \frac{C+(k+1)D}{C+kD}.$$

Thus, by induction, there is a sequence of positive integers (k_n) for $n \geqslant 1$ which satisfies $a_n/a_{n-1} = (C+k_nD)/(C+(k_n-1)D)$ for all positive integers n. Moreover, we have $k_1 = 1$ and

$$k_{n+1} = \begin{cases} k_n, & \text{if } b_n = G; \\ k_n+1, & \text{if } b_n = A. \end{cases}$$

If there are only finitely many values of n such that $b_n = A$ then the problem statement obviously holds (we can choose $d = 1$). Thus, we may assume that $b_n = A$ for infinitely many n. This means that the sequence (k_n) attains all positive integer values. Given a value $q \geqslant 1$, denote by m_q the last index where value q occurs, that is, the index such that $k_{m_q} = q$ and $k_{m_q+1} = q+1$.

Our aim is to prove that the sequence of differences $(m_{q+1} - m_q)$ is eventually constant. We first show that it is bounded above. To that end, fix $t \geqslant 1$ (we will choose a suitably large t later on) and consider a sequence $s(t)_0, s(t)_1, \ldots$ defined for $q \geqslant 1$ by $s(t)_q = a_{m_q}/(C+qD)^t$.

We note two properties of $s(t)_q$. First, simple algebra gives

$$\begin{aligned} s(t)_{q+1} &= \frac{a_{m_{q+1}}}{(C+(q+1)D)^t} = \frac{a_{m_q}}{(C+(q+1)D)^t}\left(\frac{C+(q+1)D}{C+qD}\right)^{m_{q+1}-m_q} \\ &= \frac{a_{m_q}}{(C+qD)^t}\left(\frac{C+(q+1)D}{C+qD}\right)^{m_{q+1}-m_q-t} = s(t)_q\left(\frac{C+(q+1)D}{C+qD}\right)^{m_{q+1}-m_q-t}. \end{aligned}$$

It follows that

$$\left.\begin{array}{l} s(t)_q > s(t)_{q+1} \\ s(t)_q = s(t)_{q+1} \\ s(t)_q < s(t)_{q+1} \end{array}\right\} \qquad \text{if and only if} \qquad \left\{\begin{array}{l} m_{q+1}-m_q < t, \\ m_{q+1}-m_q = t, \\ m_{q+1}-m_q > t. \end{array}\right.$$

Second, suppose that $m_{q+1}-m_q \geqslant t$ for some positive integer q. We claim that in that case $s(t)_q$ is a positive integer. Indeed, we have

$$a_{m_q+t} = a_{m_q}\left(\frac{C+(q+1)D}{C+qD}\right)^t,$$

because $k_{m_q+1} = k_{m_q+2} = \cdots = k_{m_q+t} = q+1$. Since $C+(q+1)D$ and $C+qD$ are coprime we have that

$$s(t)_q = \frac{a_{m_q}}{(C+qD)^t}$$

is an integer.

We choose $T \geqslant 1$ such that $s(T)_1 < 1$ (which exists since $C+D>1$). Then, by induction we can show that $s(T)_q < 1$ for all q. Indeed, since $s(T)_q < 1$, it is not a positive integer; this means that $m_{q+1}-m_q < T$ by the second property above. Hence by the first property above we have $s(T)_{q+1} < s(T)_q < 1$, as needed.

This means that $m_{q+1}-m_q < T$ for all q. Thus there is a largest integer $T' \leqslant T$ with the property that an equality $m_{q+1}-m_q = T'$ holds for infinitely many values of q.

Therefore, for all sufficiently large values of q we have the inequality $m_{q+1}-m_q \leqslant T'$, which by the first property implies that the sequence $s(T')$ is decreasing from some point on. Moreover, we know that the sequence attains infinitely many integer values since there are infinitely many values of q for which we have the equality $m_{q+1}-m_q = T'$. As a consequence, the sequence $s(T')$ is constant from some sufficiently large index Q onwards.

This in turn means that the equality $m_{q+1}-m_q = T'$ holds for all $q \geqslant Q$. Note that $b_n = A$ is equivalent to the fact that $n = m_q$ for some integer q. Thus, the sequence (b_n) is periodic for $n \geqslant Q$ with period T', and the proof is complete.

Solution 2. First, observe that the statement holds immediately if $b_n = G$ for all n; otherwise, there must be some n for which $b_n = A$. Without loss of generality, we may assume that $n = 1$, as we can translate the sequence without affecting the statement.

We define an arithmetic sequence (p_n) by taking $p_0 = a_0/\gcd(a_0,a_1)$ and $p_1 = a_1/\gcd(a_0,a_1)$. Note that $p_0 < p_1$, and hence that (p_n) is an increasing sequence of positive integers, and also that $p_2 = a_2/\gcd(a_0,a_1)$.

We also define a sequence of positive integers $d_n = a_n - a_{n-1}$ and a sequence of positive rational numbers $q_n = a_n/a_{n-1}$.

Then the following facts are immediate consequences of the definitions:

- if $b_n = G$, then $q_{n+1} = q_n$ and $d_{n+1} = d_n q_n$;
- if $b_n = A$, then $d_{n+1} = d_n$;

- $q_1 = p_1/p_0$;
- if $b_n = A$ and $q_n = p_i/p_{i-1}$, then $q_{n+1} = p_{i+1}/p_i$.

Now, let k_i be the number of integers n for which $b_n = G$ and $q_n = p_i/p_{i-1}$. If some k_i is infinite then b_n is eventually always G; otherwise, all values of k_i are nonnegative integers.

The sequence of values for d_n can be written as

$$d_0, d_0\frac{p_1}{p_0}, \ldots, d_0\left(\frac{p_1}{p_0}\right)^{k_1}, d_0\left(\frac{p_1}{p_0}\right)^{k_1}\frac{p_2}{p_1}, \ldots, d_0\left(\frac{p_1}{p_0}\right)^{k_1}\left(\frac{p_2}{p_1}\right)^{k_2}, \ldots$$

and in particular all terms in this sequence are positive integers. Furthermore, p_i and p_{i+1} are coprime for all i, so the following sequence consists entirely of positive integers:

$$\begin{aligned} u_0 &= d_0 p_0^{-k_1}, \\ u_1 &= d_0 p_0^{-k_1} p_1^{k_1-k_2}, \\ u_2 &= d_0 p_0^{-k_1} p_1^{k_1-k_2} p_2^{k_2-k_3}, \\ &\vdots . \end{aligned}$$

We will prove that k_i is eventually constant, which implies that the sequence of b_n is eventually periodic with period consisting of k copies of G followed by an A (where k is that constant value).

Observe that either k_i is unbounded, or is bounded with eventual maximum k for some constant k. In the second case, let r_0 be minimal such that $k_{r_0} = k$; in the first case let $r_0 = 0$. We will construct an infinite sequence of integers as follows:

- If $k_{r_i+1} \geqslant k_{r_i}$, then $r_{i+1} = r_i + 1$
- If $k_{r_i+1} < k_{r_i}$, then r_{i+1} is the minimal positive integer greater than r_i such that $k_{r_{i+1}} \geqslant k_{r_i}$.

Observe that in the second case, such an r_{i+1} must exist by our construction of r_0.

We claim that $u_{r_{i+1}} \leqslant u_{r_i}$ with equality only if $k_{r_i+1} = k_{r_i}$ (so $r_{i+1} = r_i + 1$). Indeed, if $k_{r_i+1} \geqslant k_{r_i}$ then

$$u_{r_{i+1}} = u_{r_i+1} = u_{r_i} p_{r_i}^{k_{r_i}-k_{r_i+1}} \leqslant u_{r_i},$$

with equality if and only if $k_{r_i} = k_{r_i+1}$.

Otherwise, we have

$$\frac{u_{r_{i+1}}}{u_{r_i}} = p_{r_i}^{k_{r_i}-k_{r_i+1}} p_{r_i+1}^{k_{r_i+1}-k_{r_i+2}} \cdots p_{r_{i+1}-1}^{k_{r_{i+1}-1}-k_{r_{i+1}}},$$

so we just need to show that the right hand side is strictly less than 1. But this follows because

$$\begin{aligned} p_{r_i}^{k_{r_i}-k_{r_i+1}} p_{r_i+1}^{k_{r_i+1}-k_{r_i+2}} \cdots p_{r_{i+1}-1}^{k_{r_{i+1}-1}-k_{r_{i+1}}} &< p_{r_i+1}^{k_{r_i}-k_{r_i+2}} p_{r_i+2}^{k_{r_i+2}-k_{r_i+3}} \cdots p_{r_{i+1}-1}^{k_{r_{i+1}-1}-k_{r_{i+1}}} \\ &< p_{r_i+2}^{k_{r_i}-k_{r_i+3}} p_{r_i+3}^{k_{r_i+3}-k_{r_i+4}} \cdots p_{r_{i+1}-1}^{k_{r_{i+1}-1}-k_{r_{i+1}}} \\ &\vdots \\ &< p_{r_{i+1}-1}^{k_{r_i}-k_{r_{i+1}}} \\ &\leqslant 1, \end{aligned}$$

where each inequality besides the last follows from the fact that $p_j < p_{j+1}$ and $k_{r_i} > k_j$ for $j < r_{i+1}$, and the last follows from the fact that $k_{r_i} \leqslant k_{r_{i+1}}$.

Finally, the sequence u_{r_i} is an infinite nonincreasing sequence of positive integers so must eventually be constant, yielding the claim.

Comment. The two solutions above differ in approach, but have some overlap in the structure they reveal. Indeed, the $C + nD$ of Solution 1 is the p_n of Solution 2, while the $m_{r+1} - m_r$ of Solution 1 turns out to be equal to the k_r of Solution 2.

A7. Let $\mathbb{Q}$ be the set of rational numbers. Let $f\colon \mathbb{Q} \to \mathbb{Q}$ be a function such that the following property holds: for all x, $y \in \mathbb{Q}$,

$$f(x + f(y)) = f(x) + y \qquad \text{or} \qquad f(f(x) + y) = x + f(y).$$

Determine the maximum possible number of elements of $\{f(x) + f(-x) \mid x \in \mathbb{Q}\}$.

(Japan)

Answer: 2 is the maximum number of elements.

Common remarks. Suppose that f is a function satisfying the condition of the problem. We will use the following throughout all solutions.

- $a \sim b$ if either $f(a) = b$ or $f(b) = a$,
- $a \to b$ if $f(a) = b$,
- $P(x, y)$ to denote the proposition that either $f(x + f(y)) = f(x) + y$ or $f(f(x) + y) = x + f(y)$,
- $g(x) = f(x) + f(-x)$.

With this, the condition $P(x, y)$ could be rephrased as saying that $x + f(y) \sim f(x) + y$, and we are asked to determine the maximum possible number of elements of $\{g(x) \mid x \in \mathbb{Q}\}$.

Solution 1. We begin by providing an example of a function f for which there are two values of $g(x)$. We take the function $f(x) = \lfloor x \rfloor - \{x\}$, where $\lfloor x \rfloor$ denotes the floor of x (that is, the largest integer less than or equal to x) and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

First, we show that f satisfies $P(x, y)$. Given $x, y \in \mathbb{Q}$, we have

$$\begin{aligned} f(x) + y &= \lfloor x \rfloor - \{x\} + \lfloor y \rfloor + \{y\} = (\lfloor x \rfloor + \lfloor y \rfloor) + (\{y\} - \{x\}); \\ x + f(y) &= \lfloor x \rfloor + \{x\} + \lfloor y \rfloor - \{y\} = (\lfloor x \rfloor + \lfloor y \rfloor) + (\{x\} - \{y\}). \end{aligned}$$

If $\{x\} < \{y\}$, then we have that the fractional part of $f(x) + y$ is $\{y\} - \{x\}$ and the floor is $\lfloor x \rfloor + \lfloor y \rfloor$, so $f(x) + y \to x + f(y)$. Likewise, if $\{x\} > \{y\}$, then $x + f(y) \to f(x) + y$. Finally, if $\{x\} = \{y\}$, then $f(x) + y = x + f(y) = \lfloor x \rfloor + \lfloor y \rfloor$ is an integer. In all cases, the relation P is satisfied.

Finally, we observe that if x is an integer then $g(x) = 0$, and if x is not an integer then $g(x) = -2$, so there are two values for $g(x)$ as required.

Now, we prove that there cannot be more than two values of $g(x)$. $P(x, x)$ tells us that $x + f(x) \sim x + f(x)$, or in other words, for all x,

$$f(x + f(x)) = x + f(x). \tag{1}$$

We begin with the following lemma.

Lemma 1. f is a bijection, and satisfies

$$f(-f(-x)) = x. \tag{2}$$

Proof. We first prove that f is injective. Suppose that $f(x_1) = f(x_2)$; then $P(x_1, x_2)$ tells us that $f(x_1) + x_2 \sim f(x_2) + x_1$. Without loss of generality, suppose that $f(x_1) + x_2 \to f(x_2) + x_1$.

But $f(x_1) = f(x_2)$, so $f(f(x_1) + x_2) = f(f(x_2) + x_2) = f(x_2) + x_2$ by (1). Therefore, $f(x_2) + x_1 = f(x_2) + x_2$, as required.

Now, (1) with $x = 0$ tells us that $f(f(0)) = f(0)$ and so by injectivity $f(0) = 0$.

Applying $P(x, -f(x))$ tells us that $0 \sim x + f(-f(x))$, so either $0 = f(0) = x + f(-f(x))$ or $f(x + f(-f(x))) = 0$ which implies that $x + f(-f(x)) = 0$ by injectivity. Either way, we deduce that $x = -f(-f(x))$, or $x = f(-f(-x))$ by replacing x with $-x$.

Finally, note that bijectivity follows immediately from (2). □

Since f is bijective, it has an inverse, which we denote f^{-1}. Rearranging (2) (after replacing x with $-x$) gives that $f(-x) = -f^{-1}(x)$. We have $g(x) = f(x) + f(-x) = f(x) - f^{-1}(x)$.

Suppose $g(x) = u$ and $g(y) = v$, where $u \neq v$ are both nonzero. Define $x' = f^{-1}(x)$ and $y' = f^{-1}(y)$; by definition, we have

$$\begin{aligned} x' &\to x \to x' + u \\ y' &\to y \to y' + v. \end{aligned}$$

Putting in $P(x', y)$ gives $x + y \sim x' + y' + v$, and putting in $P(x, y')$ gives $x + y \sim x' + y' + u$. These are not equal since $u \neq v$, and $x + y$ may have only one incoming and outgoing arrow because f is a bijection, so we must have either $x' + y' + u \to x + y \to x' + y' + v$ or the same with the arrows reversed. Swapping (x, u) and (y, v) if necessary, we may assume without loss of generality that this is the correct direction for the arrows.

Also, we have $-x' - u \to -x \to -x'$ by Lemma 1. Putting in $P(x + y, -x' - u)$ gives $y \sim y' + v - u$, and so $y' + v - u$ must be either $y' + v$ or y'. This means u must be either 0 or v, and this contradicts our assumption about u and v.

Comment. Lemma 1 can also be proven as follows. We start by proving that f must be surjective. Suppose not; then, there must be some t which does not appear in the output of f. $P(x, t - f(x))$ tells us that $t \sim x + f(t - f(x))$, and so by assumption $f(t) = x + f(t - f(x))$ for all x. But setting $x = f(t) - t$ gives $t = f(t - f(f(t) - t))$, contradicting our assumption about t.

Now, choose some t such that $f(t) = 0$; such a t must exist by surjectivity. $P(t, t)$ tells us that $f(t) = t$, or in other words $t = 0$ and $f(0) = 0$. The remainder of the proof is the same as the proof given in Solution 1.

Solution 2. We again start with Lemma 1, and note $f(0) = 0$ as in the proof of that lemma.

$P(x, -f(y))$ gives $x + f(-f(y)) \sim f(x) - f(y)$, and using (2) this becomes $x - y \sim f(x) - f(y)$. In other words, either $f(x - y) = f(x) - f(y)$ or $x - y = f(f(x) - f(y))$. In the latter case, we deduce that

$$\begin{aligned} f(-(x - y)) &= f(-f(f(x) - f(y))) \\ f(y - x) &= f(-f(f(x) - f(y))) \\ &= f(y) - f(x). \end{aligned}$$

Thus, $f(y) - f(x)$ is equal to either $f(y - x)$ or $-f(x - y)$. Replacing y with $x + d$, we deduce that $f(x + d) - f(x) \in \{f(d), -f(-d)\}$.

Now, we prove the following claim.

Claim. For any $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Q}$, we have that either $g(d) = 0$ or $g(d) = \pm g(d/n)$.

In particular, if $g(d/n) = 0$ then $g(d) = 0$.

Proof. We first prove that if $g(d/n) = 0$ then $g(d) = 0$. Suppose that $g(d/n) = 0$. Then $f(d/n) = -f(-d/n)$ and so $f(x + d/n) - f(x) = f(d/n)$ for any x. Applying this repeatedly, we deduce that $f(x + d) - f(x) = nf(d/n)$ for any x. Applying this with $x = 0$ and $x = -d$ and adding gives $f(d) + f(-d) = 0$, so $g(d) = 0$, and in particular the claim is true whenever $g(d) = 0$.

Now, select $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Q}$ such that $g(d) \neq 0$, and observe that we must have $g(d/n) \neq 0$. Observe that for any $k \in \mathbb{Z}$ we have that $f(kd/n) - f((k-1)d/n) \in \{f(d/n), -f(-d/n)\}$. Let A_i be the number of $k \in \mathbb{Z}$ with $i - n < k \leqslant i$ such that this difference equals $f(d/n)$.

We deduce that for any $i \in \mathbb{Z}$,

$$\begin{aligned} f(id/n) - f(id/n - d) &= \sum_{i-n<k\leqslant i} f(kd/n) - f((k-1)d/n) \\ &= A_i f(d/n) - (n - A_i) f(-d/n) \\ &= -nf(-d/n) + A_i g(d/n). \end{aligned}$$

Since $g(d/n)$ is nonzero, this is a nonconstant linear function of A_i. However, there are only two possible values for $f(id/n) - f(id/n - d)$, so there must be at most two possible values for A_i as i varies. And since $A_{i+1} - A_i \in \{-1, 0, 1\}$, those two values must differ by 1 (if there are two values).

Now, we have

$$\begin{aligned} f(d) - f(0) &= -nf(-d/n) + A_n g(d/n), \qquad \text{and} \\ f(0) - f(-d) &= -nf(-d/n) + A_0 g(d/n). \end{aligned}$$

Subtracting these (using the fact that $f(0) = 0$) we obtain

$$\begin{aligned} f(d) + f(-d) &= (A_n - A_0) g(d/n) \\ &= \pm g(d/n), \end{aligned}$$

where the last line follows from the fact that $g(d)$ is nonzero. □

It immediately follows that there can only be one nonzero number of the form $g(x)$ up to sign; to see why, if $g(d)$ and $g(d')$ are both nonzero, then for some $n, n' \in \mathbb{Z}_{>0}$ we have $d/n = d'/n'$. But

$$g(d) = \pm g(d/n) = \pm g(d').$$

Finally, suppose that for some d, d' we have $g(d) = c$ and $g(d') = -c$ for some nonzero c. So we have

$$f(d) + f(-d) - f(d') - f(-d') = 2c$$

which rearranges to become $(f(d) - f(d')) - (f(-d') - f(-d)) = 2c$.

Each of the bracketed terms must be equal to either $f(d - d')$ or $-f(d' - d)$. However, they cannot be equal since c is nonzero, so $g(d - d') = f(d - d') + f(d' - d) = \pm 2c$. This contradicts the assertion that $g(-x) = \pm c$ for all x.

Solution 3. As in Solution 1, we start by establishing Lemma 1 as above, and write $f^{-1}(x) = -f(-x)$ for the inverse of f, and $g(x) = f(x) - f^{-1}(x)$.

We now prove the following.

Lemma 2. If $g(x) \neq g(y)$, then $g(x + y) = \pm(g(x) - g(y))$.

Proof. Assume x and y are such that $g(x) \neq g(y)$. Applying $P(x, f^{-1}(y))$ gives $x + y \sim f(x) + f^{-1}(y)$, and applying $P(f^{-1}(x), y)$ gives $x + y \sim f^{-1}(x) + f(y)$.

Observe that

$$\begin{aligned}(f(x) + f^{-1}(y)) - (f^{-1}(x) + f(y)) &= (f(x) - f^{-1}(x)) - (f(y) - f^{-1}(y)) \\ &= g(x) - g(y).\end{aligned}$$

By assumption, $g(x) \neq g(y)$, and so $f(x) + f^{-1}(y) \neq f^{-1}(x) + f(y)$. Since f is bijective, this means that these two values must be $f(x + y)$ and $f^{-1}(x + y)$ in some order, and so $g(x+y) = f(x+y) - f^{-1}(x+y)$ must be their difference up to sign, which is either $g(x) - g(y)$ or $g(y) - g(x)$. □

Claim. If x and q are rational numbers such that $g(q) = 0$ and n is an integer, then $g(x + nq) = g(x)$.

Proof. If $g(b) = 0$ and $g(a) \neq g(a + b)$, then the lemma tells us that $g(b) = \pm(g(a + b) - g(a))$, which contradicts our assumptions. Therefore, $g(a) = g(a + b)$ whenever $g(b) = 0$.

A simple induction then gives that $g(nb) = 0$ for any positive integer n, and $g(nb) = 0$ for negative n as $g(x) = g(-x)$. The claim follows immediately. □

Lemma 3. There cannot be both positive and negative elements in the range of g.

Proof. Suppose that $g(x) > 0$ and $g(y) < 0$. Let $\mathcal{S}$ be the set of numbers of the form $mx + ny$ for integers m, n. We first show that $g(\mathcal{S})$ has infinitely many elements. Indeed, suppose $g(\mathcal{S})$ is finite, and let $a \in \mathcal{S}$ maximise g and $b \in \mathcal{S}$ maximise $-g$. Then $a + b \in \mathcal{S}$, and $g(a + b) = g(a) - g(b)$ or $g(b) - g(a)$. In the first case $g(a + b) > g(a)$ and in the second case $g(a + b) < g(b)$; in either case we get a contradiction.

Now, we show that there must exist some nonzero rational number q with $g(q) = 0$. Indeed, suppose first that $a + f(a) = 0$ for all a. Then $g(a) = f(a) + f(-a) = 0$ for all a, and so g takes no nonzero value. Otherwise, there is some a with $a + f(a) \neq 0$, and so (1) yields that $f(q) = 0$ for $q = a + f(a) \neq 0$. Noting that $f(-q) = 0$ from Lemma 1 tells us that $g(q) = 0$, as required.

Now, there must exist integers s and s' such that $xs = qs'$ and integers t and t' such that $yt = qt'$. The claim above gives that the value of $g(mx + ny)$ depends only on the values of m mod s and n mod t, so $g(mx + ny)$ can only take finitely many values. □

Finally, suppose that $g(x) = u$ and $g(y) = v$ where $u \neq v$ have the same sign. Assume $u, v > 0$ (the other case is similar) and assume $u > v$ without loss of generality.

$P(f^{-1}(x), f^{-1}(y))$ gives $x - y \sim f^{-1}(x) - f^{-1}(y) = f(x) - f(y) - (u - v)$, and $P(x, y)$ gives $x - y \sim f(x) - f(y)$. $u - v$ is nonzero, so $f(x - y)$ and $f^{-1}(x - y)$ must be $f(x) - f(y) - (u - v)$ and $f(x) - f(y)$ in some order, and since $g(x - y)$ must be nonnegative, we have

$$f(x) - f(y) - (u - v) \to x - y \to f(x) - f(y).$$

Then, $P(x - y, f^{-1}(y))$ tells us that $(x - y) + y \sim (f(x) - f(y)) + (f(y) - v)$, so $x \sim f(x) - v$, contradicting either $v \neq u$ or $v > 0$.

Comment. Lemma 2 also follows from $f(x + d) - f(x) \in \{f(d), -f(-d)\}$ as proven in Solution 2. Indeed, we also have $f(-x) - f(-x - d) \in \{f(d), -f(-d)\}$, and then subtracting the second from the first we get $g(x+d) - g(x) \in \{g(d), -g(d), 0\}$. Replacing $x+d$ and x with x and $-y$ gives the statement of Lemma 2.

Comment. It is possible to prove using Lemma 2 that g must have image of the form $\{0, c, 2c\}$ if it has size greater than 2. Indeed, if $g(x) = c$ and $g(y) = d$ with $0 < c < d$, then $g(x + y) = d - c$ as it must be nonnegative, and $g(y) = g((x + y) + (-x)) = |d - 2c|$ provided that $d \neq 2c$.

However, it is not possible to rule out $\{0, c, 2c\}$ based entirely on the conclusion of Lemma 2; indeed, the function given by

$$g(x) = \begin{cases} 0, & \text{if } x = 2n \text{ for } n \in \mathbb{Z}; \\ 2, & \text{if } x = 2n + 1 \text{ for } n \in \mathbb{Z}; \\ 1, & \text{if } x \notin \mathbb{Z}. \end{cases}$$

satisfies the conclusion of Lemma 2 (even though there is no function f giving this choice of g).

Note. Solution 1 actually implies that the result also holds over $\mathbb{R}$. The proposal was originally submitted and evaluated over $\mathbb{Q}$ as it is presented here, and the Problem Selection Committee believes that this form is more suitable for the competition because it allows for more varied and interesting approaches once Lemma 1 has been established. Even the variant here defined over $\mathbb{Q}$ was found to be fairly challenging.

A8. Let $p \neq q$ be coprime positive integers. Determine all infinite sequences $a_1, a_2, \ldots$ of positive integers such that the following conditions hold for all $n \geqslant 1$:

$$\max(a_n, a_{n+1}, \ldots, a_{n+p}) - \min(a_n, a_{n+1}, \ldots, a_{n+p}) = p \qquad \text{and}$$
$$\max(a_n, a_{n+1}, \ldots, a_{n+q}) - \min(a_n, a_{n+1}, \ldots, a_{n+q}) = q.$$

(Japan)

Answer: The only such sequences are $a_n = n + C$, where C is a nonnegative integer.

Common remarks.

- Denote by $a_{[i,j]}$ the subsequence $a_i, a_{i+1}, \ldots, a_j$.
- Without loss of generality, in each solution we suppose $p < q$. It can be convenient to treat the case where $p = 1$ separately.
- The problem can also be posed for sequences of arbitrary integers (rather than positive). Refer to the comment after Solution 1 for a proof.

Solution 1. Let $k = \lceil \frac{q}{p} \rceil$. Note that $k \geqslant 2$.

Lemma 1. If i, j and m are positive integers such that $|i - j| \leqslant mp$ then $|a_i - a_j| \leqslant mp$.

Proof. By the given condition, if $|i - j| \leqslant p$ then $|a_i - a_j| \leqslant p$. So the lemma follows from induction on m and the triangle inequality. □

Lemma 2. For a fixed n, suppose that a_i is minimal over $i \geqslant n$. Then $i \leqslant n + p - 1$.

Proof. Suppose for contradiction that $i \geqslant n + p$. Then $\min(a_{[i-p,i+q-p]}) = a_i$. Since $q - p \leqslant (k-1)p$, it follows from Lemma 1 that $\max(a_{[i-p,i+q-p]}) \leqslant a_i + (k-1)p < a_i + q$, which is a contradiction. □

Lemma 3. For a fixed $n > q$, suppose that a_i is maximal over $i \leqslant n$. Then $i \geqslant n - p + 1$.

Proof. Suppose a_j is minimal over $j \geqslant n - q$. Then by Lemma 2, $j \leqslant n - q + p - 1$. So $\min(a_{[n-q,n]}) = a_j$ and $a_i \geqslant \max(a_{[n-q,n]})$, which implies that $a_i \geqslant a_j + q$.

Lemma 2 also implies that if $j \geqslant n$ then $a_j \geqslant \min(a_{[n,n+p]})$. So if $i < j$, then we have $a_j \geqslant a_i - p$, which contradicts $a_i \geqslant a_j + q$. Hence we must have $i > j$.

The above inequality also gives $|a_i - a_j| \geqslant q > (k-1)p$, so by Lemma 1 it follows that $|i - j| > (k-1)p$. Therefore $i > j + (k-1)p \geqslant n - q + (k-1)p \geqslant n - p + 1$. □

Let b_n be the minimal value of a_i for $i \geqslant n$. By Lemma 2, $b_{n+p} > b_n$ for all n. Hence $b_n = \min(a_{[n,n+p]}) = \min(a_{[n,n+q]})$. Let c_n be the maximal value of a_i for $i \leqslant n$. By Lemma 3, $c_{n-p} > c_n$ for all $n > q$. Hence $c_n = \max(a_{[n-p,n]}) = \max(a_{[n-q,n]})$ for $n > q$.

So if $n > q$ then $b_n = c_{n+p} - p = c_{n+q} - q$. So for $n > q$ we get $b_{n+q-p} + p = c_{n+q} = b_n + q$, and hence $b_{n+q-p} = b_n + q - p$.

Next note that $b_{n+p} \leqslant a_{n+p} \leqslant b_n + p$. So $b_{n+p} - b_n \leqslant p$ for all $n > q$, and iterating this $(q-p)$ times gives $b_{n+p(q-p)} - b_n \leqslant p(q-p)$. But using $b_{n+q-p} = b_n + q - p$ gives $b_{n+p(q-p)} - b_n = p(q-p)$. Since equality occurs, we must have $b_{n+p} = b_n + p$.

So for $n > q$, $b_{n+p} = b_n + p$ and $b_{n+q-p} = b_n + q - p$. Since p and $q - p$ are coprime, $b_{n+1} = b_n + 1$ for all $n > q$. The only way for b_n and b_{n+1} to be different is if $b_n = a_n$, so we deduce that $a_{n+1} = a_n + 1$ and there is a constant C such that $a_n = n + C$ for all $n > q$.

Finally, suppose $a_n = n + C$ for all $n \geqslant N$. Then $p = \max(a_{N-1}, N + C + p - 1) - \min(a_{N-1}, N + C)$. So $a_{N-1} = N + C + p$ or $N + C - 1$. Similarly, $a_{N-1} = N + C + q$ or $N + C - 1$. Hence $a_{N-1} = N + C - 1$. So, by induction, we have $a_n = n + C$ for all positive integers n. Since $a_1 \geqslant 1$, C is a nonnegative integer.

It is trivial to check that $a_n = n + C$ satisfies the given condition.

Comment. Here is a variant of Solution 1. Proceed up to proving $b_n - n$ is eventually periodic with period $q-p$. Then there is some minimal value of $b_n - n$. Suppose n attains this minimal value. Since $b_{n+p} - n - p \leqslant b_n - n$, $n+p$ also attains this minimal value. And since p and $q-p$ are coprime, all $n \geqslant q$ must attain this minimal value. Hence $b_{n+1} = b_n + 1$ for all $n \geqslant q$. Finish as above.

Comment. It is also possible to solve the problem using a weaker version of Lemma 2 and without Lemma 3. For example, the following lemma plays a similar role.

Lemma 2'. Let $b'_n = \min(a_{[n,n+p]})$. Then $b'_n < b'_{n+p}$.

Comment. To solve the problem for sequences a_n of arbitrary integers, we will use the following lemma.

Lemma 4. The sequence a_n is either bounded above or bounded below.

Proof. Suppose that a_n is unbounded above and below. Then there is some i such that $a_i < a_1 - p$. There is also some j such that $a_j > \max(a_{[1,i]}) + q$. Now let a_l be minimal amongst $a_{[1,j]}$. Since $a_l \leqslant a_i$, $a_l < a_1 - p$ and $a_l < a_j - kp$. By Lemma 1, $1 + p < l < j - kp$. So $\min(a_{[l-p,l+q-p]}) = a_l$. By Lemma 1 again, $\max(a_{[l-p,l+q-p]}) \leqslant a_l + (k-1)p < a_l + q$, which is a contradiction. □

From there, the solution above can be adapted to prove that $a_n = n + C$ for all n or $a_n = -n + C$ for all n, where C can be any constant integer.

Solution 2. For $n, x \geqslant 1$, let the *x-width* of n be $\max(a_{[n,n+x]}) - \min(a_{[n,n+x]})$. We call a positive integer x *good* if the x-width of n is less than or equal to x for all sufficiently large n, and we call x *very good* if the x-width of n is equal to x for sufficiently large n.

Lemma 1. If p' is good and q' is very good with $p' < q' < 2p'$, then $2p' - q'$ is also good.

Proof. Note that $0 < q' - p' < p' < q'$. Let n be a sufficiently large positive integer. Then for $k \in [n + q' - p', n + p']$, we have $a_k \geqslant \max(a_{[n,n+p']}) - p'$ and $a_k \geqslant \max(a_{[n+q'-p',n+q']}) - p'$ since p' is good, which shows $a_k \geqslant \max(a_{[n,n+q']}) - p'$. Similarly we get $a_k \leqslant \min(a_{[n,n+q']}) + p'$.

Therefore, for all $k \in [n + q' - p', n + p']$ we have $a_k \in [\max(a_{[n,n+q']}) - p', \min(a_{[n,n+q']}) + p']$. Thus, the $(2p' - q')$-width of $n + q' - p'$ is at most $(\min(a_{[n,n+q']}) + p') - (\max(a_{[n,n+q']}) - p') = 2p' - q'$. The lemma follows. □

Lemma 2. Let p' be a good number and q' a very good number with $p' < q'$. For sufficiently large n, take $s, t \in [n, n + q']$ such that $\min(a_{[n,n+q']}) = a_s$ and $\max(a_{[n,n+q']}) = a_t$. Then $s \in [n, n + p']$ and $t \in [n + q' - p', n + q']$.

Proof. Lemma 2 and Lemma 3 from Solution 1 hold with p and q replaced by p' and q' by similar arguments. We can deduce the statement about s from Lemma 2 of Solution 1. We can deduce the statement about t from Lemma 3 of Solution 1. □

Lemma 3. If p' is good and q' is very good with $2p' < q'$, then there exists a positive integer r such that for all sufficiently large n, we have $a_{n+r} - a_n \geqslant r$.

Proof. Let $r = q' - 2p'$, and let s and t be as defined in Lemma 2. Then consider the identity

$$(a_t - a_{n+q'-p'}) + (a_{n+p'+r} - a_{n+p'}) + (a_{n+p'} - a_s) = a_t - a_s = q'.$$

By Lemma 2, we have $s \in [n, n + p']$ and $t \in [n + q' - p', n + q']$, so $a_{n+p'} - s \leqslant p'$ and $a_t - a_{n+q'-p'} \leqslant p'$. Combining these, we get $a_{n+p'+r} - a_{n+p'} \geqslant q' - 2p' = r$. This proves that $a_{n+r} - a_n \geqslant r$ for sufficiently large n. □

Lemma 4. Suppose $(p, q) \neq (1, 2)$. Then there exists a good number p' such that $2p' < q$.

Proof. Let p' be the smallest good positive integer. Note that p is good, so p' exists and is less than q.

Suppose for contradiction that $2p' \geqslant q$. If $2p' > q$, then by Lemma 1, $2p' - q$ is a good number strictly less than p', which contradicts minimality of p'. If $2p' = q$, then $p' < p < 2p'$. So we can apply Lemma 1 with $q_0 = p$ to get that $2p' - p$ is a good number that is strictly less than p', which again contradicts minimality. □

If $(p, q) = (1, 2)$ then the problem is easily solved. Otherwise, Lemmas 3 and 4 combined give us some $r > 0$ such that $a_{n+r} - a_n \geqslant r$ for n sufficiently large.

By iterating, we get $a_{n+pr} - a_n \geqslant pr$ for all sufficiently large n, and hence it follows that $a_{n+p} - a_n = p$. Similarly we get $a_{n+q} - a_n = q$. As p and q are coprime, we deduce that $a_{n+1} - a_n = 1$ for sufficiently large n. Thus we get $a_n = n + C$ for sufficiently large n, and we can conclude by the same argument as Solution 1.

Combinatorics

C1. Let n be a positive integer. A class of n students run n races, in each of which they are ranked with no draws. A student is eligible for a rating (a, b) for positive integers a and b if they come in the top b places in at least a of the races. Their final score is the maximum possible value of $a - b$ across all ratings for which they are eligible.

Find the maximum possible sum of all the scores of the n students.

(Australia)

Answer: The maximum possible sum is $\frac{n(n-1)}{2}$.

Solution 1. The answer can be achieved by the students finishing in the same order in every race. To show that this is the maximum, we will apply a series of modifications to the results of the races, each of which does not decrease the total score, such that after k such modifications the first k positions are the same in every race. Say that a student is scored on the b^{th} place if their score is $a - b$ because they came in the top b places in a of the races and b is minimal with this property for that student.

Supposing that the first $k - 1$ positions are the same in every race, look at the students scored on the k^{th} place. If there are no such students, let $\ell > k$ be minimal such that some student S is scored on the ℓ^{th} place. Then, in every race where S appears in any place from the k^{th} through the ℓ^{th} inclusive (of which there must be at least ℓ, otherwise S would achieve a higher rating of 0 based on the n^{th} place), reorder the students in places k through ℓ so that S finishes in the k^{th} place instead (and otherwise the ordering of those students is arbitrary). Now S is scored on the k^{th} place, their score has gone up by $\ell - k$ and no other scores have gone down (some might have gone up as well).

Now we know that the first $k - 1$ positions are the same in every race and at least one student is scored on the k^{th} place. Pick one such student S. In each race where S finishes behind the k^{th} place, swap them with the student T who finishes in the k^{th} place, leaving the positions of all other students unchanged. Each such swap increases the score of S by 1 and decreases the score of T by at most 1, so such swaps do not decrease the total score. At the end of this process, the first k positions are the same in every race and the total score has not decreased.

Repeating this n times yields the required result.

Comment. The following simpler approach to modifying results of races is tempting: find pairs of students S and T who are scored on places k and ℓ respectively, where $k < \ell$, but where S finishes after T in some race, and swap the positions of those two students in that race so they finish in the same order as the places they are scored on. However, such a swap can decrease the total score; for example, suppose that $k = 1$ and $\ell = 4$, and in some race S finishes 6^{th} and T finishes 3^{rd}; then swapping those students reduces the number of races contributing to T's score without increasing the number contributing to S's score.

Solution 2. The answer can be achieved by having the same ranking for all n races.

Note that taking $a = b = n$ shows each student has a nonnegative score. Consider a student who has race ranks $r_1, r_2, \ldots, r_n$ and a final score of s. We first prove that

$$\sum_i r_i \leqslant n(n - s).$$

Without loss of generality, suppose that $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_n$. There must exist some k with $s + 1 \leqslant k \leqslant n$ and $k - r_k = s$. In order to maximise $\sum_i r_i$ while retaining the score of s, we can replace each of $r_1, \ldots, r_{k-1}$ by r_k, and replace each of $r_{k+1}, \ldots, r_n$ by n. Then the sum is

$$\sum_i r_i \leqslant kr_k + (n-k)n = n^2 - k(n-r_k) = n^2 - k(n+s-k) \leqslant n^2 - s_n. \tag{1}$$

The final inequality follows from the fact that given $s+1 \leqslant k \leqslant n$, the quantity $k(n+s-k)$ is minimised when $k = n$.

The sum of ranks of all students across all races is $\frac{n^2(n+1)}{2}$. If the total of all student scores is t, then (1) implies

$$\frac{n^2(n+1)}{2} \leqslant n^3 - tn.$$

This rearranges to $t \leqslant \frac{n(n-1)}{2}$, as required.

Solution 3. In each race, assign the student in the k^{th} place a weight of $1 - \frac{k}{n}$. If a student finishes in the top b places in at least a of the races, the total of their weights is at least $a\left(1 - \frac{b}{n}\right) = a - b\left(\frac{a}{n}\right) \geqslant a - b$. Thus the sum of a student's weights across all races is at least their score, and so the sum of all weights for all students across all races is at least the sum of all the scores of all students. The sum of weights in each race is $\frac{n-1}{2}$, so the sum of all weights across all races is $\frac{n(n-1)}{2}$. Equality is achieved if and only if, for each student, the values of b and a determining that student's score have $a = n$ and they finish in exactly the b^{th} place in all n races; that is, if the students are ranked the same in every race.

Solution 4. Given a positive integer $b(S)$ for each student S, define $a_b(S)$ to be the number of races in which S finished in the top $b(S)$ places, and define $\operatorname{score}_b(S) = a_b(S) - b(S)$; for a race r, let $I_b(S, r)$ be 1 if S finished in the top $b(S)$ places in race r and 0 otherwise, so

$$a_b(S) = \sum_r I_b(S, r).$$

Then the problem asks for the maximum across all possible results of the races of

$$\max_b \sum_S \operatorname{score}_b(S) = \max_b \left(\sum_r \sum_S I_b(S, r) - \sum_S b(S) \right).$$

Given b, the sum $\sum_S I_b(S, r)$ is maximised (not necessarily uniquely) for some choice of the rankings in race r, which is the same choice for every race. So the maximum possible sum of the scores of all the students occurs when all students are ranked the same in all races, which yields the given answer.

C2. Let n be a positive integer. The integers $1, 2, 3, \ldots, n^2$ are to be written in the cells of an $n \times n$ board such that each integer is written in exactly one cell and each cell contains exactly one integer. For every integer d with $d \mid n$, the *d-division* of the board is the division of the board into $(n/d)^2$ nonoverlapping sub-boards, each of size $d \times d$, such that each cell is contained in exactly one $d \times d$ sub-board.

We say that n is a *cool number* if the integers can be written on the $n \times n$ board such that, for each integer d with $d \mid n$ and $1 < d < n$, in the d-division of the board, the sum of the integers written in each $d \times d$ sub-board is not a multiple of d.

Determine all even cool numbers.

(Türkiye)

Answer: The even cool numbers are $n = 2^k$ where k is a positive integer.

Solution. We first show by induction that $n = 2^k$ is a cool number. The base case of $n = 2$ is trivial as there is no such d.

For induction, assume that 2^k is a cool number. We construct a numbering of a $2^{k+1} \times 2^{k+1}$ board that satisfies the conditions.

Take the $2^{k+1} \times 2^{k+1}$ board and divide it into four $2^k \times 2^k$ sub-boards. By assumption, there is some numbering P of a $2^k \times 2^k$ board that satisfies the required condition; we write down the numbering P in each sub-board. Next, add 2^{2k} to every number in the second sub-board, add 2×2^{2k} to every number in the third sub-board, and add 3×2^{2k} to every number in the fourth sub-board. Then the numbers in the cells of the $2^{k+1} \times 2^{k+1}$ board are the numbers 1 to $2^{2(k+1)}$.

Now locate 2^{2k} from the first sub-board, and swap it with $2^{2k} + 2^{k-1}$ from the second sub-board. Locate 3×2^{2k} from the third sub-board, and swap it with $3 \times 2^{2k} + 2^{k-1}$ from the fourth sub-board.

We claim that this numbering of the $2^{k+1} \times 2^{k+1}$ board satisfies the required conditions. For any $d = 2^i$ where $i < k$, consider any $2^i \times 2^i$ sub-board. The sum of its cells modulo 2^i is not changed in the addition step or the swapping step, so the sum is congruent modulo 2^i to the sum of the corresponding $2^i \times 2^i$ sub-board in P, which is nonzero, as required.

In the case of $d = 2^k$, we can directly evaluate the sum of the $(b+1)^{\text{th}}$ sub-board for $b \in \{0, 1, 2, 3\}$. The sum is given by

$$2^{2k-1}(1 + 2^{2k}) + b2^{4k} + (-1)^b 2^{k-1} \equiv 2^{k-1} \pmod{2^k}.$$

Therefore all sub-boards satisfy the required conditions and so 2^{k+1} is a cool number, completing the induction.

It remains to show that no other even number is a cool number. Let $n = 2^s m$ where s is a positive integer and m is an odd integer greater than 1. For the sake of contradiction, suppose that there is a numbering of the $n \times n$ board satisfying the required conditions.

Claim. In the 2^i-division of the board, where $1 \leqslant i \leqslant s$, the sum of numbers in each $2^i \times 2^i$ sub-board is congruent to 2^{i-1} modulo 2^i.

Proof. We prove the claim by induction on i. The base case of $i = 1$ holds as the sum of numbers in each 2×2 sub-board must be odd. Next, suppose the claim is true for 2^i. In the 2^{i+1}-division, each $2^{i+1} \times 2^{i+1}$ sub-board is made up of four $2^i \times 2^i$ sub-boards, each with a sum congruent to 2^{i-1} modulo 2^i. Hence the sum of each $2^{i+1} \times 2^{i+1}$ sub-board is a multiple of 2^i. It cannot be a multiple of 2^{i+1} because of the conditions, which means it must be congruent to 2^i modulo 2^{i+1}. This proves the claim. □

Back to the problem, since m is odd, summing up the m^2 sums of $2^s \times 2^s$ sub-boards gives

$$2^{s-1}m^2 \equiv 2^{s-1} \pmod{2^s}.$$

However, the sum of the numbers from 1 to n^2 is

$$\frac{n^2(n^2+1)}{2} = 2^{2s-1}m^2(2^{2s}m^2+1) \equiv 0 \pmod{2^s}.$$

This is a contradiction. Therefore n is not a cool number.

Comment. In the case of odd n, similar arguments show that prime powers are cool numbers.

If the definition of cool numbers additionally requires that all $d \times d$ sub-boards in the d-division have the same nonzero residue modulo d, then the cool numbers are precisely the prime powers.

C3. Let n be a positive integer. There are $2n$ knights sitting at a round table. They consist of n pairs of partners, each pair of which wishes to shake hands. A pair can shake hands only when next to each other. Every minute, one pair of adjacent knights swaps places.

Find the minimum number of exchanges of adjacent knights such that, regardless of the initial arrangement, every knight can meet her partner and shake hands at some time.

(Belarus)

Answer: The minimum number of exchanges is $\frac{n(n-1)}{2}$.

Common remarks. The solution is divided into three lemmas. We provide multiple proofs of each lemma.

Solution. Join each pair of knights with a chord across the table. We'll refer to these chords as *chains*.

First we show that $n(n-1)/2$ exchanges are required for some arrangements.

Lemma 1. If each knight is initially sitting directly opposite her partner, then at least $n(n-1)/2$ exchanges are required for all knights to meet and shake hands with their partners.

Proof 1. In this arrangement any two chains are initially intersecting. For two knights to be adjacent to each other, it is necessary that their chain does not cross any other chain, and thus every pair of chains must be uncrossed at some time. Each exchange of adjacent knights can only uncross a single pair of intersecting chains, and thus the number of exchanges required is at least the number of pairs of chains, which is $n(n-1)/2$. □

Proof 2. In this arrangement the two knights in each pair are initially separated by $n-1$ seats in either direction around the table, and so each pair must move a total of at least $n-1$ steps so as to be adjacent. There are n pairs, and each exchange moves two knights by a single step. Hence at least $n(n-1)/2$ moves are required. □

We will now prove that $n(n-1)/2$ exchanges is sufficient in all cases. We'll prove a stronger version of this bound than is required, namely that every knight can shake hands with her partner at the end, after all exchanges have finished.

Begin by adding a pillar at the centre of the table. For each chain that passes through the centre of the table, we arbitrarily choose one side of the chain and say that the pillar lies on that side of the chain. While the pillar may lie on a chain, we will never move a knight if that causes the pillar to cross to the other side of a chain. Say that a chain passes in front of a knight if it passes between that knight and the pillar, and define the length of a chain to be the number of knights it passes in front of. Then each chain has a length between 0 and $n-1$ inclusive.

Say that a chain C *encloses* another chain C' if C and C' do not cross, and C passes between C' and the pillar. Say that two chains are *intersecting* if they cross on the table; *enclosing* if one chain encloses the other; and *disjoint* otherwise. Let k, l and m denote respectively the number of enclosing, intersecting and disjoint pairs of chains. Then we have

$$k+l+m=\frac{n(n-1)}{2}.$$

Lemma 2. $2k + l$ exchanges are sufficient to reach a position with all pairs of knights sitting adjacent to each other.

Proof 1. We proceed by induction on $2k + l$.

If every chain has length 0, then every pair of knights is adjacent and the statement is trivial.

Otherwise, let A and B be a pair of knights whose chain C_0 has length $q \geqslant 1$. Let $S_0 = A$, and let $S_1, \ldots, S_q$ be the knights which C_0 passes in front of, sitting in that order from A to B. We know that C_0 passes in front of S_1, and there are three cases for the chain C_1 for knight S_1.

If C_1 passes in front of S_0 then C_0 and C_1 are intersecting, and we can make them disjoint by exchanging the positions of S_0 and S_1. This reduces the sum $2k + l$ by 1.

If C_1 passes in front of neither S_0 nor B then C_1 is enclosed by C_0, and we can swap S_0 and S_1 to make C_0 and C_1 an intersecting pair. This increases l by 1 and decreases k by 1, and hence reduces the sum $2k + l$ by 1.

If this C_1 passes in front of B then we cannot immediately find a beneficial exchange.

In the third case, we look instead at the knights S_i and S_{i+1}, for each i in turn. Each time, we will either find a beneficial exchange, or find that the chain C_{i+1} for knight S_{i+1} passes in front of B. Eventually we will either find a beneficial exchange in one of the first two cases above, or we will find that the chain C_q for S_q passes in front of B, in which case C_q and C_0 are intersecting and we can make C_q and C_0 disjoint by swapping S_q and B.

Also note that the only times a chain is increased in length is when it is enclosed by another chain. But this cannot happen for a chain containing the pillar, so no chains ever cross the pillar. □

Proof 2. We begin by ignoring the seats, and let each knight walk freely to a predetermined destination. Each pair of knights will walk around the table to one of the two points on the circumference midway between their initial locations, such that the chain between them passes between the pillar and the destination. If more than one pair of knights would have the same destination point, then we make small adjustments to the destination points so that each pair has a distinct destination point.

We then imagine each knight walking at a constant speed (which may be different for each knight). They all start and stop walking at the same time. We want to count how many times two knights pass (either in opposite directions, or in the same direction but at different speeds). For any two pairs of knights, the number of passes depends on the relation between their two chains.

If their two chains are intersecting then there will be one pass, involving the two knights for whom the other chain passes between them and the pillar.

If their two chains are enclosing then there will be two passes, with one of the knights with the enclosing chain passing both of the knights with the shorter enclosed chain.

If their two chains are disjoint then there will be no passes.

The number of passes is therefore $2k + l$. If multiple pairs of knights would pass at the same time, we can slightly adjust the walking speeds so that the passes happen at distinct times. We can then convert this sequence of passes into a sequence of seat exchanges in the original problem, which shows that $2k + l$ exchanges is sufficient. □

Lemma 3. $k \leqslant m$.

Proof 1. We proceed by induction on n. The base case $n = 2$ is clear.

Consider a chain C of greatest length, and suppose it joins knights A and B. Let x be the number of chains that intersect C, and let y be the number of chains that are enclosed by C. Note that no chain can enclose C. Then C passes in front of one knight from each pair whose chain intersects C, and both knights in any pair whose chain is enclosed by C. Thus the length of C is $x + 2y \leqslant n - 1$. The number of chains that form a disjoint pair with C is then

$$n - 1 - x - y \geqslant (x + 2y) - x - y = y.$$

Now we can remove A and B and use the induction hypothesis. We need to show that the length of each remaining chain is at most $n - 2$ so the chains remain valid. No chain increases in length after removing A and B. If any chain C had length $n - 1$, then the chain between A and B also had length $n - 1$. Then C must have passed in front of exactly one of A or B, and so has length $n - 2$ after removing A and B. □

Proof 2. Let k_C denote the number of chains C' such that C encloses C'.

Note that if C encloses C', then $k_{C'} < k_C$.

First we will show that there at least k_C chains that are disjoint from C. Let x be the length of C, let $\mathcal{S}$ be the set of x knights that C passes in front of, and let $\mathcal{T}$ be the set of x knights sitting directly opposite them. None of the knights in $\mathcal{T}$ can have a chain that encloses or is enclosed by C, and if any knight in $\mathcal{T}$ has a chain that intersects C, then her partner must be a knight in $\mathcal{S}$. So we have that

$$\begin{aligned}
2k_C &= \text{number of knights in } \mathcal{S} \text{ whose chain is enclosed by } C \\
&= x - \text{number of knights in } \mathcal{S} \text{ whose chain intersects } C \\
&\leqslant x - \text{number of knights in } \mathcal{T} \text{ whose chain intersects } C \\
&\leqslant \text{number of knights in } \mathcal{T} \text{ whose chain is disjoint from } C \\
&\leqslant 2 \times \text{number of chains that are disjoint from } C.
\end{aligned}$$

Now let m_C denote the number of chains C' with C and C' disjoint, and $k_{C'} < k_C$. We will show that $m_C \geqslant k_C$.

Let $\mathcal{R}$ be a set of k_C chains that are disjoint from C, such that $\sum_{C' \in \mathcal{R}} k_{C'}$ is minimal. If every chain $C' \in \mathcal{R}$ has $k_{C'} < k_C$, then we are done. Otherwise, let consider a chain C' with $k_{C'} \geqslant k_C$. There are then at least k_C chains C'' for which the chain C' passes between C'' and the pillar. Each of these chains must have $k_{C''} < k_{C'}$, and at least one of them is not in $\mathcal{R}$ (otherwise $\mathcal{R}$ would contain C' and at least k_C other chains), so we can swap this chain with C' to obtain a set $\mathcal{R}'$ with $\sum_{C' \in \mathcal{R}'} k_{C'} < \sum_{C' \in \mathcal{R}} k_{C'}$. But this contradicts the minimality of $\mathcal{R}$.

We finish by summing these inequalities over all chains C:

$$k = \sum_C k_C \leqslant \sum_C m_C \leqslant m. \qquad \square$$

By Lemma 3, we have that $2k + l \leqslant k + l + m = n(n-1)/2$. Combining this with Lemma 2, we have that $n(n-1)/2$ exchanges is enough to reach an arrangement where every knight is sitting next to her partner, as desired.

Comment. Either proof of Lemma 3 can be adapted to show that the configuration in Lemma 1 is the only one which achieves the bound.

C4. On a board with 2024 rows and 2023 columns, Turbo the snail tries to move from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then moves one step at a time to an adjacent cell sharing a common side. He wins if he reaches any cell in the last row. However, there are 2022 predetermined, hidden monsters in 2022 of the cells, one in each row except the first and last rows, such that no two monsters share the same column. If Turbo unfortunately reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move.

Suppose Turbo is allowed to take n attempts. Determine the minimum value of n for which he has a strategy that guarantees reaching the last row, regardless of the locations of the monsters.

(Hong Kong)

Comment. One of the main difficulties of solving this question is in determining the correct expression for n. Students may spend a long time attempting to prove bounds for the wrong value for n before finding better strategies.

Students may incorrectly assume that Turbo is not allowed to backtrack to squares he has already visited within a single attempt. Fortunately, making this assumption does not change the answer to the problem, though it may make it slightly harder to find a winning strategy.

Answer: The answer is $n = 3$.

Solution. First we demonstrate that there is no winning strategy if Turbo has 2 attempts.

Suppose that $(2, i)$ is the first cell in the second row that Turbo reaches on his first attempt. There can be a monster in this cell, in which case Turbo must return to the first row immediately, and he cannot have reached any other cells past the first row.

Next, suppose that $(3, j)$ is the first cell in the third row that Turbo reaches on his second attempt. Turbo must have moved to this cell from $(2, j)$, so we know $j \neq i$. So it is possible that there is a monster on $(3, j)$, in which case Turbo also fails on his second attempt. Therefore Turbo cannot guarantee to reach the last row in 2 attempts.

Next, we exhibit a strategy for $n = 3$. On the first attempt, Turbo travels along the path

$$(1, 1) \to (2, 1) \to (2, 2) \to \cdots \to (2, 2023).$$

This path meets every cell in the second row, so Turbo will find the monster in row 2 and his attempt will end.

If the monster in the second row is not on the edge of the board (that is, it is in cell $(2, i)$ with $2 \leqslant i \leqslant 2022$), then Turbo takes the following two paths in his second and third attempts:

$$(1, i-1) \to (2, i-1) \to (3, i-1) \to (3, i) \to (4, i) \to \cdots \to (2024, i).$$
$$(1, i+1) \to (2, i+1) \to (3, i+1) \to (3, i) \to (4, i) \to \cdots \to (2024, i).$$

The only cells that may contain monsters in either of these paths are $(3, i-1)$ and $(3, i+1)$. At most one of these can contain a monster, so at least one of the two paths will be successful.

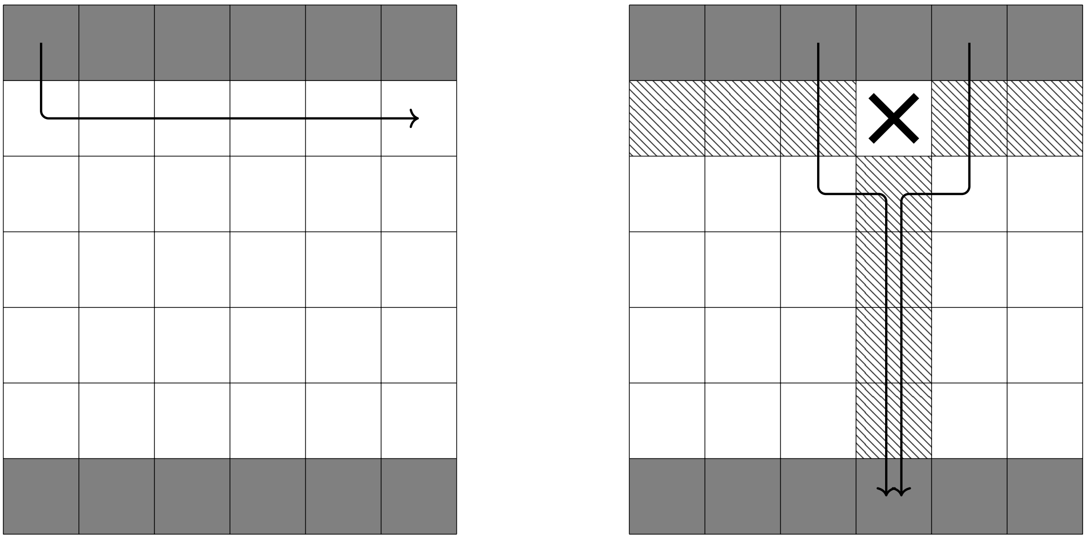


Figure 1: Turbo's first attempt, and his second and third attempts in the case where the monster on the second row is not on the edge. The cross indicates the location of a monster, and the shaded cells are cells guaranteed to not contain a monster.

If the monster in the second row is on the edge of the board, without loss of generality we may assume it is in $(2,1)$. Then, on the second attempt, Turbo takes the following path:

$$(1,2) \to (2,2) \to (2,3) \to (3,3) \to \cdots \to (2022,2023) \to (2023,2023) \to (2024,2023).$$

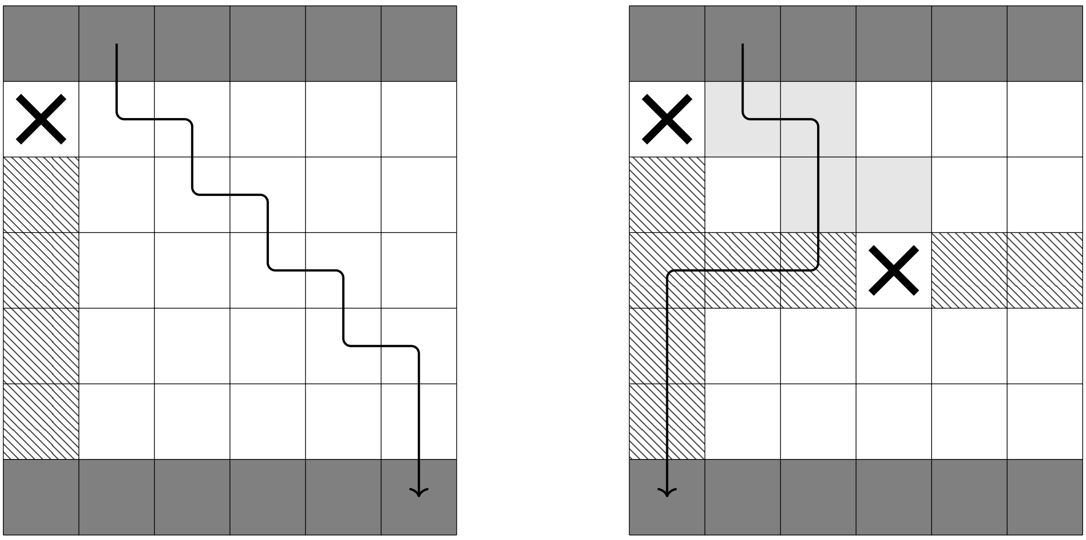


Figure 2: Turbo's second and third attempts in the case where the monster on the second row is on the edge. The light gray cells on the right diagram indicate cells that were visited on the previous attempt. Note that not all safe cells have been shaded.

If there are no monsters on this path, then Turbo wins. Otherwise, let (i,j) be the first cell on which Turbo encounters a monster. We have that $j = i$ or $j = i+1$. Then, on the third attempt, Turbo takes the following path:

$$\begin{aligned}(1,2) &\to (2,2) \to (2,3) \to (3,3) \to \cdots \to (i-2,i-1) \to (i-1,i-1)\\ &\to (i,i-1) \to (i,i-2) \to \cdots \to (i,2) \to (i,1)\\ &\to (i+1,1) \to \cdots \to (2023,1) \to (2024,1).\end{aligned}$$

Now note that

- The cells from $(1,2)$ to $(i-1,i-1)$ do not contain monsters because they were reached earlier than (i,j) on the previous attempt.
- The cells (i,k) for $1 \leqslant k \leqslant i-1$ do not contain monsters because there is only one monster in row i, and it lies in (i,i) or $(i,i+1)$.
- The cells $(k,1)$ for $i \leqslant k \leqslant 2024$ do not contain monsters because there is at most one monster in column 1, and it lies in $(2,1)$.

Therefore Turbo will win on the third attempt.

Comment. A small variation on Turbo's strategy when the monster on the second row is on the edge is possible. On the second attempt, Turbo can instead take the path

$$\begin{aligned}(1,2023) &\to (2,2023) \to (2,2022) \to \cdots \to (2,3) \to (2,2) \to (2,3) \to \cdots \to (2,2023)\\ &\to (3,2023) \to (3,2022) \to \cdots \to (3,4) \to (3,3) \to (3,4) \to \cdots \to (3,2023)\\ &\to \cdots\\ &\to (2022,2023) \to (2022,2022) \to (2022,2023)\\ &\to (2023,2023)\\ &\to (2024,2023).\end{aligned}$$

If there is a monster on this path, say in cell (i,j), then on the third attempt Turbo can travel straight down to the cell just left of the monster instead of following the path traced out in the second attempt.

$$\begin{aligned}(1,j-1) &\to (2,j-1) \to \cdots \to (i-1,j-1) \to (i,j-1)\\ &\to (i,j-2) \to \cdots \to (i,2) \to (i,1)\\ &\to (i+1,1) \to \cdots \to (2023,1) \to (2024,1).\end{aligned}$$

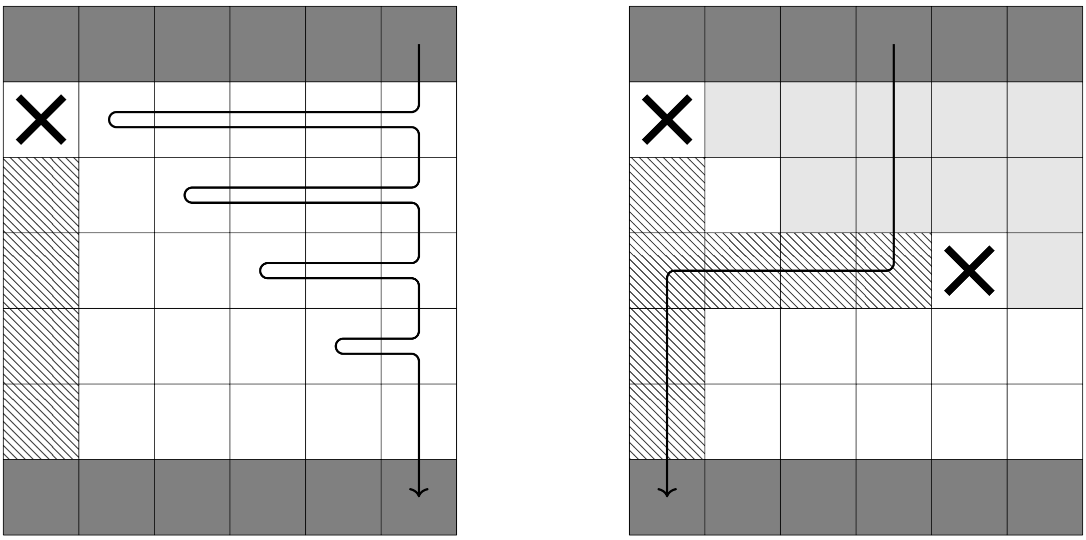


Figure 3: Alternative strategy for Turbo's second and third attempts.

C5. Let N be a positive integer. Geoff and Ceri play a game in which they start by writing the numbers $1, 2, \ldots, N$ on a board. They then take turns to make a move, starting with Geoff. Each move consists of choosing a pair of integers (k, n), where $k \geqslant 0$ and n is one of the integers on the board, and then erasing every integer s on the board such that $2^k \mid n - s$. The game continues until the board is empty. The player who erases the last integer on the board loses.

Determine all values of N for which Geoff can ensure that he wins, no matter how Ceri plays.

(Indonesia)

Answer: The answer is that Geoff wins when N is of the form 2^n for n odd or of the form $t2^n$ for n even and $t > 1$ odd.

Common remarks. We will say that a set $\mathcal{S}$ wins if the current player wins given $\mathcal{S}$ as the current set of integers on the board. Otherwise, we will say that $\mathcal{S}$ loses.

We will let $J(\mathcal{S}, \mathcal{T}) = (2\mathcal{S} - 1) \cup (2\mathcal{T})$. Note that every subset of $\mathbb{Z}$ can be written as $J(\mathcal{S}, \mathcal{T})$ for some unique pair $(\mathcal{S}, \mathcal{T})$ of subsets of $\mathbb{Z}$.

We will let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Solution.

Lemma 1. For any set $\mathcal{S}$, $\mathcal{S}$ wins if and only if $J(\mathcal{S}, \varnothing)$ wins. Similarly, $\mathcal{S}$ wins if and only if $J(\varnothing, \mathcal{S})$ wins.

Proof. Let (k, m) be a move on $\mathcal{S}$, and let $\mathcal{T}$ be the result of applying the move. Then we can reduce $J(\mathcal{S}, \varnothing)$ to $J(\mathcal{T}, \varnothing)$ by applying the move $(k + 1, 2m - 1)$.

Conversely, let (k, m) be a move on $J(\mathcal{S}, \varnothing)$. We can express the result of this move as $J(\mathcal{T}, \varnothing)$ for some $\mathcal{T}$. Then we can reduce $\mathcal{S}$ to $\mathcal{T}$ by applying the move $(\max(k-1, 0), (k+1)/2)$.

This gives us a natural bijection between games starting with $\mathcal{S}$ and games starting with $J(\mathcal{S}, \varnothing)$ and thus proves the first part of the lemma. The second part follows by a similar argument. □

Lemma 2. If $\mathcal{S}$ and $\mathcal{T}$ are nonempty and at least one of them loses, then $J(\mathcal{S}, \mathcal{T})$ wins.

Proof. If $\mathcal{S}$ is losing, then we can delete $J(\varnothing, \mathcal{T})$ using the move $(1, t)$ for some $t \in J(\varnothing, \mathcal{T})$, which leaves the losing set $J(\mathcal{S}, \varnothing)$. Similarly, if $\mathcal{T}$ is losing, then we can delete $J(\mathcal{S}, \varnothing)$ using the move $(1, s)$ for some $s \in J(\mathcal{S}, \varnothing)$, leaving the losing set $J(\varnothing, \mathcal{T})$. □

Lemma 3. If $\mathcal{S}$ is nonempty and wins, then $J(\mathcal{S}, \mathcal{S})$ loses.

Proof. From this position, we can convert any sequence of moves into another valid sequence of moves by replacing $(k, 2n-1)$ with $(k, 2n)$, and vice versa. Thus we may assume that the initial move (k, m) has m odd. We want to show that any such move results in a winning position for the other player.

The move $(0, m)$ loses immediately. Otherwise, the move results in the set $J(\mathcal{T}, \mathcal{S})$ for some set $\mathcal{T}$. There are three cases.

If $\mathcal{T}$ is empty then the other player gets the winning set $J(\varnothing, \mathcal{S})$.

If $\mathcal{T}$ is losing then the other player can choose the move $(1, s)$ for some $s \in J(\varnothing, \mathcal{S})$, which leaves the losing set $J(\mathcal{T}, \varnothing)$.

If $\mathcal{T}$ is nonempty winning then the other player can choose the move $(k, m + 1)$, which results in the position $J(\mathcal{T}, \mathcal{T})$. We can then proceed by induction on $|\mathcal{S}|$ to show that this is a losing set. □

Lemma 4. $[2n]$ wins if and only if $[n]$ loses.

Proof. Note that $[2n] = J([n],[n])$. The result then follows directly from the previous two lemmas. □

Lemma 5. For any integer $n \geqslant 1$, $[2n+1]$ wins.

Proof. By Lemma 4, either $[n]$ or $[2n]$ loses. If $[n]$ loses, then by Lemma 2 we have that $[2n+1] = J([n+1],[n])$ wins. Otherwise, $[2n]$ loses, and therefore $[2n+1]$ wins by choosing the move $(k, 2n+1)$ for sufficiently large k so that only $2n+1$ is eliminated. □

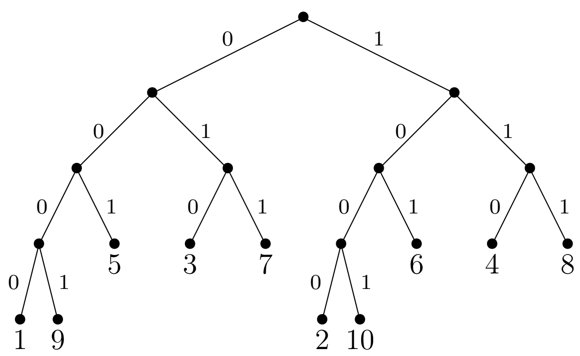
It remains to verify the original answer. We have two cases to consider:

- Suppose $N = 2^n$ for some n. For $N = 1$, every move is an instant loss for Geoff. Then by Lemma 4, Geoff wins for $N = 2^n$ if and only if Geoff loses for $N = 2^{n-1}$, and thus by induction we have that Geoff wins for $N = 2^n$ if and only if n is odd.

- Otherwise, $N = t2^n$, for some n and some $t > 1$ with t odd. By Lemma 5, Geoff wins when $n = 0$. Then by Lemma 4, Geoff wins for $N = t2^n$ if and only if Geoff loses for $N = t2^{n-1}$, and thus by induction on n we have that Geoff wins for $N = t2^n$ if and only if n is even.

Comment. We can represent this game as a game on partial binary trees. This representation could be common in rough working, as it facilitates exploration of small cases. If two sets produce trees which are topologically equivalent, then this equivalence leads to a natural bijection between games starting with the two sets. Such equivalences lead to a significant reduction in the number of distinct cases that need to be considered when exploring the game for small N.

The construction is as follows. First we begin by considering an infinite binary tree. For each positive integer n, we consider the binary representation of $n-1$, starting with the least significant bit and ending with an infinite sequence of leading zeroes. We map this sequence of bits to a path on the binary tree by starting at the root, and then repeatedly choosing the left child if the bit is 0 and the right child if the bit is 1. We can then truncate each path after reaching a sufficient depth to distinguish the path from all other paths in the tree.



Valid moves in this representation of the game consist of selecting a node with two children, and removing either the left child or the right child (and its descendants). Selecting and removing the entire graph is also an allowed move (which loses instantly).

Two trees have equivalent games if they're topologically identical. This equivalence includes swapping the left and right children of any single node, or removing a node with a single child by merging the edges above and below it (and decreasing the depth of its children by one).

Comment. We can also analyse this game using Grundy values (also known as nim-values or nimbers). This requires a slight modification to the rules, wherein any move that would erase all integers on the board is disallowed, and the first player who cannot move loses. This is clearly equivalent to the original game.

Let $g(\mathcal{S})$ denote the Grundy value of the game starting with the set $\mathcal{S}$. Note that the bijection in Lemma 1 shows that

$$g(\mathcal{S}) = g(J(\mathcal{S}, \varnothing)) = g(J(\varnothing, \mathcal{S})).$$

For any set V, let $\operatorname{mex}(V)$ denote the least nonnegative element that is not an element of V. For nonnegative integers x and y, define $j(x, y)$ recursively as

$$j(x, y) = \operatorname{mex}(\{x, y\} \cup \{j(w, y) \mid w < x\} \cup \{j(x, z) \mid z < y\}).$$

The values of $j(x, y)$ for small x and y are:

5	6	7	8	9	1	0
4	5	3	6	2	0	1
3	4	5	1	0	2	9
2	3	4	0	1	6	8
1	2	0	4	5	3	7
0	1	2	3	4	5	6
—	0	1	2	3	4	5

We can show that $g(J(\mathcal{S}, \mathcal{T})) = j(g(\mathcal{S}), g(\mathcal{T}))$ for any nonempty sets $\mathcal{S}$ and $\mathcal{T}$. The remainder of the proof follows a similar structure to the given solution.

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C6. Let n and T be positive integers. James has $4n$ marbles with weights $1, 2, \ldots, 4n$. He places them on a balance scale, so that both sides have equal weight. Andrew may move a marble from one side of the scale to the other, so that the absolute difference in weights of the two sides remains at most T.

Find, in terms of n, the minimum positive integer T such that Andrew may make a sequence of moves such that each marble ends up on the opposite side of the scale, regardless of how James initially placed the marbles.

(*Ghana*)

Answer: The minimum value of T is $4n$.

Solution 1. We must have $T \geqslant 4n$, as otherwise we can never move the marble of weight $4n$. We will show that $T = 4n$ by showing that, for any initial configuration, there is a sequence of moves, never increasing the absolute value of the difference above $4n$, that results in every marble ending up on the opposite side of the scale. Because moves are reversible, it suffices to do the following: exhibit at least one configuration C for which this can be achieved, and show that any initial configuration can reach such a configuration C by some sequence of moves.

Consider partitioning the weights into pairs $(t, 4n+1-t)$. Suppose that each side of the balance contains n of those pairs. If one side of the balance contains the pair $(t, 4n+1-t)$ for $1 \leqslant t < 2n$ and the other side contains $(2n, 2n+1)$, then the following sequence of moves swaps those pairs between the sides without ever increasing the absolute value of the difference above $4n$.

$$t, 4n+1-t \mid 2n, 2n+1 \tag{1}$$
$$t, 2n, 4n+1-t \mid 2n+1 \tag{2}$$
$$t, 2n \mid 2n+1, 4n+1-t \tag{3}$$
$$t, 2n, 2n+1 \mid 4n+1-t \tag{4}$$
$$2n, 2n+1 \mid t, 4n+1-t \tag{5}$$

Applying this sequence twice swaps any two pairs $(t, 4n+1-t)$ and $(t', 4n+1-t')$ between the sides. So we can achieve an arbitrary exchange of pairs between the sides, and C can be any configuration where each side of the balance contains n of those pairs.

We now show that any initial configuration can reach one where each side has n of those pairs. Consider a configuration where one side has total weight $A-s$ and the other has total weight $A+s$, for some $0 \leqslant s \leqslant 2n$, and where some pair is split between the two sides. (If no pair is split between the two sides, they must have equal weights and we are done.) Valid moves include moving any weight w with $1 \leqslant w \leqslant 2n+s$ from the $A+s$ side to the $A-s$ side, and moving any weight w with $1 \leqslant w \leqslant 2n-s$ from the $A-s$ side to the $A+s$ side. Suppose the pair $(t, 4n+1-t)$, with $t \leqslant 2n$, is split between the sides. If t is on the $A+s$ side, or on the $A-s$ side and $t \leqslant 2n-s$, it can be moved to the other side. Otherwise, t is on the $A-s$ side and $t \geqslant 2n-s+1$, so $4n+1-t \leqslant 2n+s$ is on the $A+s$ side and can be moved to the other side. So we can unite the two weights from that pair without splitting any other pair, and repeating this we reach a configuration where no pair is split between the sides.

Solution 2. As in Solution 1, $T \geqslant 4n$. Let δ be the weight of the left side minus the weight of the right side. A configuration is called *legal* if $|\delta| \leqslant 4n$, and a move is *legal* if it results in a legal configuration. We will show that if $\delta = 0$ then there is a sequence of legal moves after which every marble is on the opposite side.

We treat the $n = 1$ case separately. The initial configuration has marbles 1, 4 on one side and 2, 3 on the other. So moving marbles 2, 4, 3, 1 in that order is legal and every marble ends on the opposite side. Now assume $n \geqslant 2$.

Marbles of weight at most $2n$ are called *small*. We will make use of the following lemmas:

Lemma 1. If a pair of legal configurations differ only in the locations of small marbles then there is a sequence of legal moves to get from one to the other.

Proof. At first we only move marbles in the wrong position if they are not on the lighter side. (In the case of a tie, neither side is lighter.) Such a move is always legal. Since this reduces the number of marbles in the wrong position, eventually it will no longer be possible to perform such a move.

Then the only marbles in the wrong position are on the lighter side. So moving one marble in the wrong position at a time will always increase $|\delta|$, and $|\delta| \leqslant 4n$ at the end. Hence every move is legal. □

Lemma 2. Let $k \in \mathbb{N}$. A positive integer can be expressed as a sum of distinct positive integers up to k if and only if it is at most $k(k+1)/2$.

Proof. The maximum possible sum of distinct positive integers up to k is $k(k+1)/2$. For the other direction we use induction on k. The case $k = 1$ is trivial. Assume the statement is true for $k-1$. For positive integers up to k we only need a single term. For larger integers, including k in the expression means we are done by the inductive hypothesis. □

Also note that $n(2n+1) \geqslant 4n$ for $n \geqslant 2$.

Let $2n < m \leqslant 4n$. Marbles of weight greater than m are called *big* and marbles from $2n+1$ to m are called *medium*.

Suppose all big marbles are on the correct side (that is, opposite where they started), m is on the incorrect side and the configuration is legal. Then the following steps give a sequence of legal moves after which m is on the correct side and the big marbles were never moved.

Assume m is on the left. In Step 2, we rearrange the small marbles so we can move m. But this is only possible if the weight of big and medium marbles on the right is not too large. So we may need to move some medium marbles from the right first, which we do in Step 1.

Step 1 Skip to Step 2 if the total weight of medium and big marbles on the right side is at most $n(4n+1) + 2n - m$. Since the big marbles are in the correct position and m is in the incorrect position, the big marbles on the right can weigh at most $n(4n+1) - m$. So there must be a medium marble $m' < m$ on the right.

From the first assumption, it is legal to move all small marbles to the left. Then by Lemma 2 we can move some of the small marbles to the right so the right side has weight exactly $n(4n+1) + 2n$. Then moving m' is legal. Repeat this step. Since the total weight of medium marbles on the right decreases, this step will occur a bounded number of times.

Step 2 Let the total weight of the right side be $n(4n+1) + 2n - m + x$ and the weight of small marbles on the right side be y. Note that $y \geqslant x$. If $x \leqslant 0$ then moving m is legal.

Otherwise, by Lemma 2 there is a set of small marbles of weight $y - x$. By Lemma 1, there is a sequence of legal moves of small marbles such that the right side has weight exactly $n(4n+1) + 2n - m$. Now moving m is legal.

Applying the process above for $m = 4n, 4n-1, \ldots, 2n+1$ will move all nonsmall marbles to the opposite side. Then Lemma 1 completes the proof.

C7. Let N be a positive integer and let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Suppose that, for each $n > N$, a_n is equal to the number of times a_{n-1} appears in the list $a_1, a_2, \ldots, a_{n-1}$.

Prove that at least one of the sequences $a_1, a_3, a_5, \ldots$ and $a_2, a_4, a_6, \ldots$ is eventually periodic.

(Australia)

Solution 1. Let $M > \max(a_1, \ldots, a_N)$. We first prove that some integer appears infinitely many times. If not, then the sequence contains arbitrarily large integers. The first time each integer larger than M appears, it is followed by a 1. So 1 appears infinitely many times, which is a contradiction.

Now we prove that every integer $x \geqslant M$ appears at most $M-1$ times. If not, consider the first time that any $x \geqslant M$ appears for the M^{th} time. Up to this point, each appearance of x is preceded by an integer which has appeared $x \geqslant M$ times. So there must have been at least M numbers that have already appeared at least M times before x does, which is a contradiction.

Thus there are only finitely many numbers that appear infinitely many times. Let the largest of these be k. Since k appears infinitely many times there must be infinitely many integers greater than M which appear at least k times in the sequence, so each integer $1, 2, \ldots, k-1$ also appears infinitely many times. Since $k+1$ doesn't appear infinitely often there must only be finitely many numbers which appear more than k times. Let the largest such number be $l \geqslant k$. From here on we call an integer x *big* if $x > l$, *medium* if $l \geqslant x > k$ and *small* if $x \leqslant k$. To summarise, each small number appears infinitely many times in the sequence, while each big number appears at most k times in the sequence.

Choose a large enough $N' > N$ such that $a_{N'}$ is small, and in $a_1, \ldots, a_{N'}$:

- every medium number has already made all of its appearances;
- every small number has made more than $\max(k, N)$ appearances.

Since every small number has appeared more than k times, past this point each small number must be followed by a big number. Also, by definition each big number appears at most k times, so it must be followed by a small number. Hence the sequence alternates between big and small numbers after $a_{N'}$.

Lemma 1. Let g be a big number that appears after $a_{N'}$. If g is followed by the small number h, then h equals the amount of small numbers which have appeared at least g times before that point.

Proof. By the definition of N', the small number immediately preceding g has appeared more than $\max(k, N)$ times, so $g > \max(k, N)$. And since $g > N$, the g^{th} appearance of every small number must occur after a_N and hence is followed by g. Since there are k small numbers and g appears at most k times, g must appear exactly k times, always following a small number after a_N. Hence on the h^{th} appearance of g, exactly h small numbers have appeared at least g times before that point. □

Denote by $a_{[i,j]}$ the subsequence $a_i, a_{i+1}, \ldots, a_j$.

Lemma 2. Suppose that i and j satisfy the following conditions:

(a) $j > i > N' + 2$,

(b) a_i is small and $a_i = a_j$,

(c) no small value appears more than once in $a_{[i,j-1]}$.

Then a_{i-2} is equal to some small number in $a_{[i,j-1]}$.

Proof. Let $\mathcal{I}$ be the set of small numbers that appear at least a_{i-1} times in $a_{[1,i-1]}$. By Lemma 1, $a_i = |\mathcal{I}|$. Similarly, let $\mathcal{J}$ be the set of small numbers that appear at least a_{j-1} times in $a_{[1,j-1]}$. Then by Lemma 1, $a_j = |\mathcal{J}|$ and hence by (b), $|\mathcal{I}| = |\mathcal{J}|$. Also by definition, $a_{i-2} \in \mathcal{I}$ and $a_{j-2} \in \mathcal{J}$.

Suppose the small number a_{j-2} is not in $\mathcal{I}$. This means a_{j-2} has appeared less than a_{i-1} times in $a_{[1,i-1]}$. By (c), a_{j-2} has appeared at most a_{i-1} times in $a_{[1,j-1]}$, hence $a_{j-1} \leqslant a_{i-1}$. Combining with $a_{[1,i-1]} \subset a_{[1,j-1]}$, this implies $\mathcal{I} \subseteq \mathcal{J}$. But since $a_{j-2} \in \mathcal{J} \setminus \mathcal{I}$, this contradicts $|\mathcal{I}| = |\mathcal{J}|$. So $a_{j-2} \in \mathcal{I}$, which means it has appeared at least a_{i-1} times in $a_{[1,i-1]}$ and one more time in $a_{[i,j-1]}$. Therefore $a_{j-1} > a_{i-1}$.

By (c), any small number appearing at least a_{j-1} times in $a_{[1,j-1]}$ has also appeared $a_{j-1}-1 \geqslant a_{i-1}$ times in $a_{[1,i-1]}$. So $\mathcal{J} \subseteq \mathcal{I}$ and hence $\mathcal{I} = \mathcal{J}$. Therefore, $a_{i-2} \in \mathcal{J}$, so it must appear at least $a_{j-1} - a_{i-1} = 1$ more time in $a_{[i,j-1]}$. □

For each small number a_n with $n > N' + 2$, let p_n be the smallest number such that $a_{n+p_n} = a_i$ is also small for some i with $n \leqslant i < n + p_n$. In other words, $a_{n+p_n} = a_i$ is the first small number to occur twice after a_{n-1}. If $i > n$, Lemma 2 (with $j = n + p_n$) implies that a_{i-2} appears again before a_{n+p_n}, contradicting the minimality of p_n. So $i = n$. Lemma 2 also implies that $p_n \geqslant p_{n-2}$. So $p_n, p_{n+2}, p_{n+4}, \ldots$ is a nondecreasing sequence bounded above by $2k$ (as there are only k small numbers). Therefore, $p_n, p_{n+2}, p_{n+4}, \ldots$ is eventually constant and the subsequence of small numbers is eventually periodic with period at most k.

Note. Since every small number appears infinitely often, Solution 1 additionally proves that the sequence of small numbers has period k. The repeating part of the sequence of small numbers is thus a permutation of the integers from 1 to k. It can be shown that every permutation of the integers from 1 to k is attainable in this way.

Solution 2. We follow Solution 1 until after Lemma 1. For each $n > N'$ we keep track of how many times each of 1, 2, …, k has appeared in $a_1, \dots, a_n$. We will record this information in an updating $(k+1)$-tuple

$$(b_1, b_2, \dots, b_k; j)$$

where each b_i records the number of times i has appeared. The final element j of the $(k+1)$-tuple, also called the *active* element, represents the latest small number that has appeared in $a_1, \dots, a_n$.

As n increases, the value of $(b_1, b_2, \dots, b_k; j)$ is updated whenever a_n is small. The $(k+1)$-tuple updates deterministically based on its previous value. In particular, when $a_n = j$ is small, the active element is updated to j and we increment b_j by 1. The next big number is $a_{n+1} = b_j$. By Lemma 1, the next value of the active element, or the next small number a_{n+2}, is given by the number of b terms greater than or equal to the newly updated b_j, or

$$|\{i \mid 1 \leqslant i \leqslant k,\ b_i \geqslant b_j\}|. \tag{1}$$

Each sufficiently large integer which appears $i+1$ times must also appear i times, with both of these appearances occurring after the initial block of N. So there exists a global constant C such that $b_{i+1} - b_i \leqslant C$. Suppose that for some r, $b_{r+1} - b_r$ is unbounded from below. Since the value of $b_{r+1} - b_r$ changes by at most 1 when it is updated, there must be some update where $b_{r+1} - b_r$ decreases and $b_{r+1} - b_r < -(k-1)C$. Combining with the fact that $b_i - b_{i-1} \leqslant C$ for all i, we see that at this particular point, by the triangle inequality

$$\min(b_1, \dots, b_r) > \max(b_{r+1}, \dots, b_k). \tag{2}$$

Since $b_{r+1} - b_r$ just decreased, the new active element is r. From this point on, if the new active element is at most r, by (1) and (2), the next element to increase is once again from $b_1, \dots, b_r$. Thus only $b_1, \dots, b_r$ will increase from this point onwards, and b_k will no longer increase, contradicting the fact that k must appear infinitely often in the sequence. Therefore $|b_{r+1} - b_r|$ is bounded.

Since $|b_{r+1} - b_r|$ is bounded, it follows that each of $|b_i - b_1|$ is bounded for $i = 1, \dots, k$. This means that there are only finitely many different states for $(b_1 - b_1, b_2 - b_1, \dots, b_k - b_1; j)$. Since the next active element is completely determined by the relative sizes of $b_1, b_2, \dots, b_k$ to each other, and the update of b terms depends on the active element, the active element must be eventually periodic. Therefore the small numbers subsequence, which is either $a_1, a_3, a_5, \dots$ or $a_2, a_4, a_6, \dots$, must be eventually periodic.

C8. Let n be a positive integer. Given an $n \times n$ board, the unit cell in the top left corner is initially coloured black, and the other cells are coloured white. We then apply a series of colouring operations to the board. In each operation, we choose a 2×2 square with exactly one cell coloured black and we colour the remaining three cells of that 2×2 square black.

Determine all values of n such that we can colour the whole board black.

(Peru)

Answer: The answer is $n = 2^k$ where k is a nonnegative integer.

Solution 1. We first prove by induction that it is possible the colour the whole board black for $n = 2^k$. The base case of $k = 1$ is trivial. Assume the result holds for $k = m$ and consider the case of $k = m + 1$. Divide the $2^{m+1} \times 2^{m+1}$ board into four $2^m \times 2^m$ sub-boards. Colour the top left $2^m \times 2^m$ sub-board using the inductive hypothesis. Next, colour the centre 2×2 square with a single operation. Finally, each of the remaining $2^m \times 2^m$ sub-board can be completely coloured using the inductive hypothesis, starting from the black square closest to the centre. This concludes the induction.

Now we prove that if such a colouring is possible for n then n must be a power of 2. Suppose it is possible to colour an $n \times n$ board where $n > 1$. Identify the top left corner of the board by $(0, 0)$ and the bottom right corner by (n, n). Whenever an operation takes place in a 2×2 square centred on (i, j), we immediately draw an "X", joining the four cells' centres (see Figure 4). Also, identify this X by (i, j). The first operation implies there's an X at $(1, 1)$. Since the whole board is eventually coloured, every cell centre must be connected to at least one X. The collection of all Xs forms a graph G.

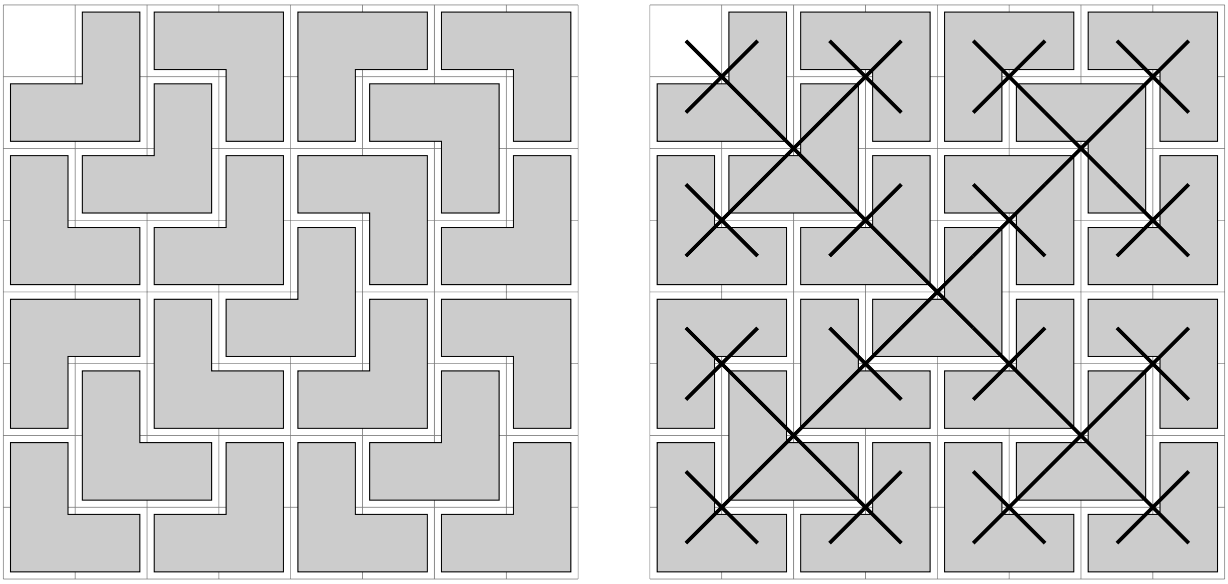


Figure 4: L-trominoes placements corresponding to colouring operations (left) and the corresponding X diagram (right).

Claim 1. The graph G is a tree.

Proof. Since every operation requires a pre-existing black cell, each newly drawn X apart from the first must connect to an existing X. So all Xs are connected to the first X and G must be connected. Now, suppose G has a cycle. Consider the newest X involved in the cycle, it must connect to previous Xs at at least two points. But this implies the corresponding operation will colour at most two cells, which is a contradiction. □

Note that in the following arguments, Claims 2 to 4 only require the condition that G is a tree and every cell is connected to G.

Claim 2. If there's an X at (i, j), then $1 \leqslant i, j \leqslant n-1$ and $i \equiv j \pmod 2$.

Proof. The inequalities $1 \leqslant i, j \leqslant n-1$ are clear. Call an X at (i, j) *good* if $i \equiv j \pmod 2$, or *bad* if $i \not\equiv j \pmod 2$. The first X at $(i, j) = (1, 1)$ is good. Suppose some Xs are bad. Since G is connected, there must exist a good X connecting to a bad X. But this can only occur if they connect at two points, creating a cycle. This is a contradiction, thus all Xs are good. □

Call an X at (i, j) *odd* if $i \equiv j \equiv 1 \pmod 2$, *even* if $i \equiv j \equiv 0 \pmod 2$.

Claim 3. The integer n must be even. Furthermore, there must be $4(n/2-1)$ odd Xs connecting the cells on the perimeter of the board as shown in Figure 5.

Proof. If n is odd, the four corners of the bottom left cell are $(n, 0), (n-1, 0), (n-1, 1)$ and $(n, 1)$, none of which satisfies the conditions of Claim 2. So the bottom left cell cannot connect to any X. If n is even, each cell on the edge of the board has exactly one corner satisfying the conditions of Claim 2, so the X connecting it is uniquely determined. Therefore the cells on the perimeter of the board are connected to Xs according to Figure 5. □

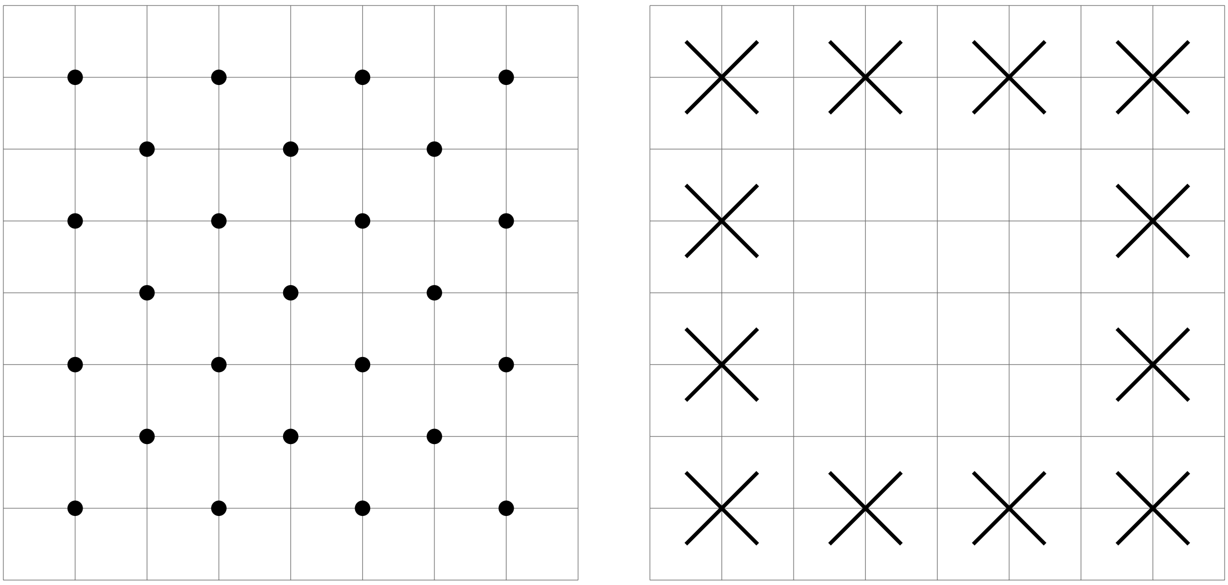


Figure 5: Highlighting the permitted points for Xs (left) and Xs on the perimeter (right).

Divide the $n \times n$ board into $n^2/4$ blocks of 2×2 squares. Call each of these blocks a *big-cell.* We say a big-cell is *filled* if it contains an odd X on its interior, *empty* otherwise. By Claim 3, each big-cell on the perimeter must be filled.

Claim 4. Every big-cell is filled.

Proof. Recall that Xs can only be at (i, j) with $i \equiv j \pmod 2$. Suppose a big-cell centred at (i, j) is empty. Then in order for its four cells to be coloured, there must be four even Xs on $(i-1, j-1)$, $(i+1, j-1)$, $(i-1, j+1)$ and $(i+1, j+1)$, "surrounding" the big-cell (see Figure 6).

By Claim 3, no empty big-cell can be on the perimeter. So if there exist some empty big-cells, the boundary between empty and filled big-cells must consist of a number of closed loops. Each closed loop is made up of several line segments of length 2, each of which separates a filled big-cell from an empty big-cell.

Since every empty big-cell is surrounded by even Xs and every filled big-cell contains an odd X, the two end points of each such line segment must be connected by Xs. Since these line segments form at least one closed loop, it implies the existence of a cycle made up of Xs (see Figure 6). This is a contradiction, thus no big-cell can be empty. □

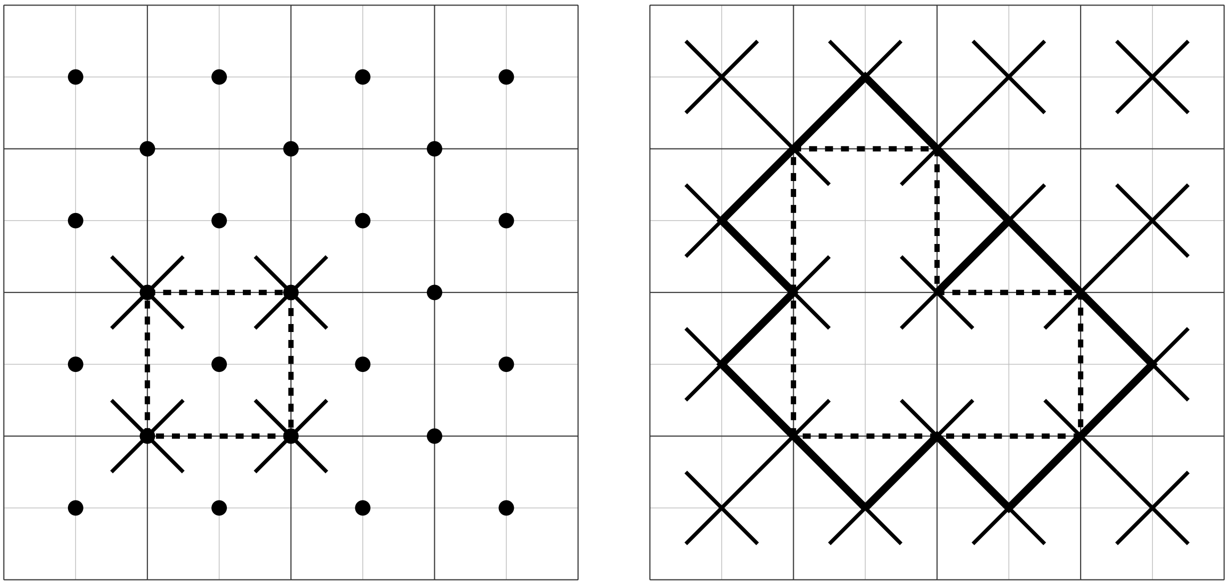


Figure 6: An empty big-cell surrounded by even Xs (left) and the boundary between empty and filled Xs creating a cycle (right).

Therefore every big-cell is filled by an odd X, and the connections between them are provided by even Xs. We can now reduce the $n \times n$ problem to an $n/2 \times n/2$ problem in the following way. Perform a dilation of the board by a factor of $1/2$ with respect to $(0, 0)$. Each big-cell is shrunk to a regular cell. For the Xs, replace each odd X at (i, j) by the point $(i/2, j/2)$, and replace each even X at (i, j) by an X at $(i/2, j/2)$.

We claim the new resulting graph of Xs is a tree that connects all cells of an $n/2 \times n/2$ board. First, two connected Xs in the original $n \times n$ board are still connected after their replacements (noting that some Xs have been replaced by single points). For each cell in the $n/2 \times n/2$ board, its centre corresponds to an odd X from a filled big-cell in the original $n \times n$ board, so it must be connected to the graph. Finally, suppose there exists a cycle in the new graph. The cycle consists of Xs that correspond to even Xs in the original graph connecting big-cells, forming a cycle of big-cells. Since in every big-cell, the four unit squares were connected by an odd X, this implies the existence of a cycle in the original graph, which is a contradiction.

Thus the new graph of Xs must be a tree that connects all cells of an $n/2 \times n/2$ board, which are the required conditions for Claims 2 to 4. Hence we can repeat our argument, halving the dimensions of the board each time, until we reach the base case of a 1×1 board (where the tree is a single point). Therefore n must be a power of 2, completing the solution.

Solution 2. As in Solution 1, it is possible the colour the whole board black for $n = 2^k$.

The colouring operation is equivalent to the placement of L-trominoes. For each L-tromino we place on the board, we draw an arrow and a node as shown in Figure 7. We also draw a node in the top left corner of the board.

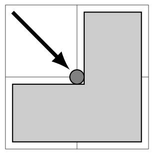


Figure 7: Tromino with corresponding arrow and node drawn.

Claim 1. The arrows and nodes form a directed tree rooted at the top left corner.

Proof. The proof is similar to the proof of Claim 1 in Solution 1, with the additional note that the directions of the arrows inherit the order of the colouring operations, so they must be pointing away from the top left node. □

Note that since all edges of the tree are diagonal, the nodes can only lie on points (i, j) with $i + j \equiv 0 \pmod 2$. This implies that we can only place down L-trominoes of one particular parity: that is, with the centre of the L-tromino on a point with $i + j \equiv 0 \pmod 2$. In the remainder of the proof, we will implicitly use this parity property when determining possible positions of L-trominoes.

Next, we show that certain configurations of edges of the tree are impossible.

Claim 2. There cannot be two edges in a "parallel" configuration (see Figure 8).

Proof. In such a configuration, the two edges can either be directed in the same direction or opposite directions. If they point in the same direction (see Figure 8), then the L-trominoes corresponding to the two edges overlap.

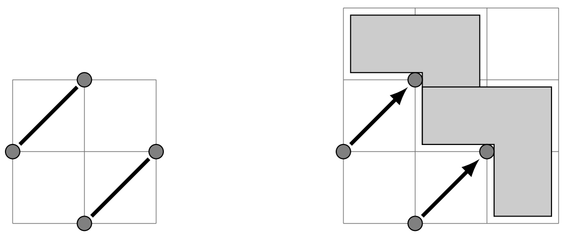


Figure 8: Parallel configuration (left) and two parallel edges, case 1 (right).

If they point in opposite directions, then we get the diagram in Figure 9. The cells marked ($\star$) must lie inside the $n \times n$ board, so they must be covered by L-trominoes. There is only one possible way to cover these with a L-tromino of the right parity. But this makes the arrows form a cycle, which cannot happen. So we have a contradiction. □

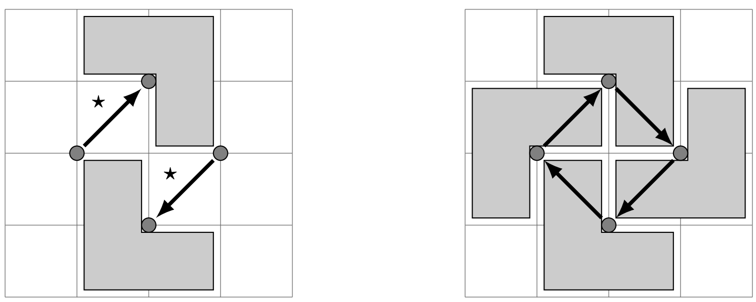


Figure 9: Two parallel edges, case 2.

Claim 3. There cannot be three edges in a "zigzag" configuration, shown in Figure 10.

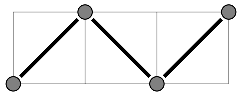


Figure 10: Zigzag configuration.

Proof. Assume for contradiction that there is a zigzag. Then take the zigzag with maximal distance from the root of the tree (measured by distance along the graph from the root to the middle edge of the zigzag).

We may assume without loss of generality that the middle edge is directed down-right. Then the right edge must be directed up-right, since no two arrows can point to the same node. Next, we draw in the corresponding L-trominoes, and consider the cell marked ($\star$). There are two possible ways to cover it with an L-tromino, because of the parity of L-tromino centres.

We could choose the centre of the L-tromino to be the top right corner of the cell (see Figure 11). This immediately gives another zigzag.

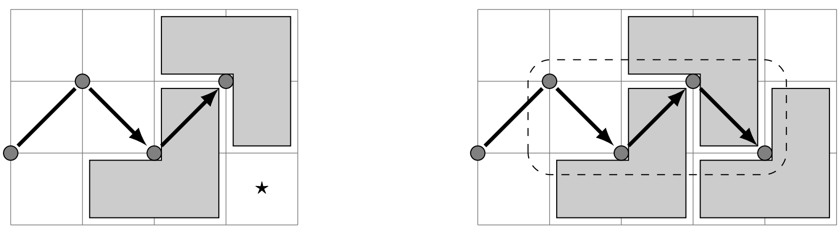


Figure 11: Zigzag configuration, case 1.

The other possibility is if we choose the centre of the L-tromino to be the bottom left corner of the cell (see Figure 12). Then we need to cover the cell marked ($\star\star$) with an L-tromino. If

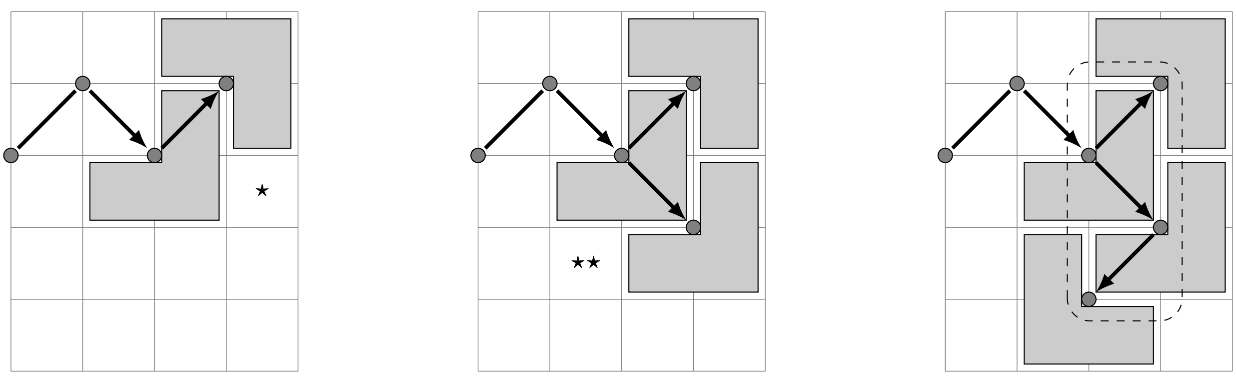


Figure 12: Zigzag configuration, case 2.

we placed the centre of the L-tromino on the top left corner of the cell, this would give two parallel edges, contradicting Claim 2. So we must place the centre of the L-tromino on the bottom right corner of the cell, which gives a zigzag.

In each case, we get another zigzag further away from the root of the tree, which contradicts our assumption of maximality. So there cannot be any zigzags. □

We now colour the nodes of the tree. Colour the root node yellow. For all other nodes, we colour it white if it has an arrow coming out of it in a different direction to the arrow going in, and black otherwise.

Claim 4. Any child of a black node is white.

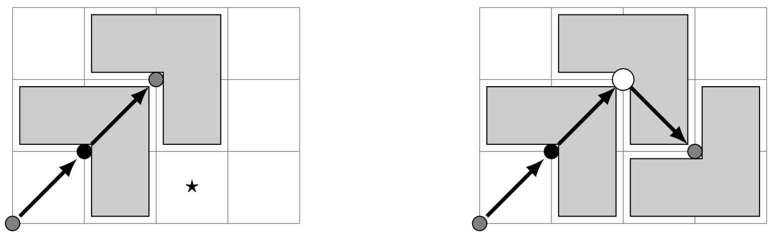


Figure 13: Black node configuration.

Proof. Suppose we have a black node with a child. Then the arrow exiting the black node must be in the same direction as the arrow entering it by the definition of our colouring, giving the left diagram of Figure 13.

The cell marked (⋆) must be covered by an L-tromino. If the centre of this L-tromino is the bottom left corner, then this would give an arrow leaving the black node in a different direction, which cannot happen. So the centre of the L-tromino must instead be the top right corner, which gives an arrow leaving the upper node in a different direction. Thus the upper node must be white. □

Claim 5. Every white node has three children, all of which are black.

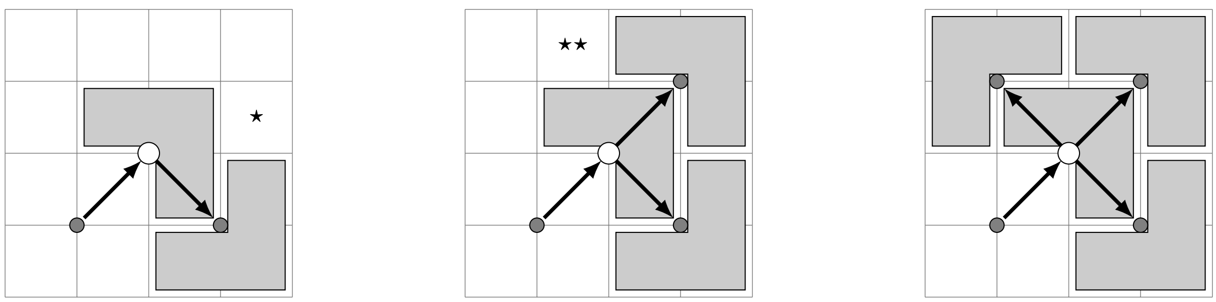


Figure 14: White node configuration.

Proof. Refer to Figure 14. Suppose we have a white node, as in the leftmost diagram. The cell marked (⋆) must be covered by an L-tromino. If the centre of this L-tromino is the bottom right corner of the cell, then this would form a zigzag, which by Claim 3 is not allowed. So the centre must be the top left corner.

Next, the cell marked (⋆⋆) must be covered by an L-tromino. If the centre of this L-tromino is the top right corner, this would form a zigzag, so the centre must be the bottom left corner instead. Thus we have shown that any white node has three children.

Finally, note that if any of the child nodes had three children of their own, then this would give parallel edges in the diagram, which contradicts Claim 2. Therefore the child nodes of the white node must all be black. □

We now know that the node colours alternate between black and white as you go down the tree, so all white nodes lie on points with coordinates $(2i, 2j)$, and all black nodes lie on points with coordinates $(2i + 1, 2j + 1)$.

Now (assuming $n > 1$) we will construct a new board whose cells are 2×2 squares of our current board. We replace the root node and its child with a single big cell and a big root node,

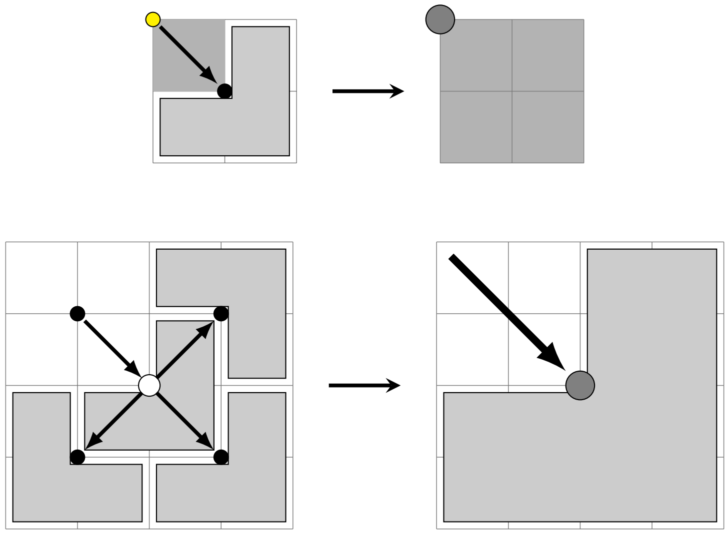


Figure 15: Replacing with larger cells and L-trominoes.

and we replace each white node and its three children with a big L-tromino, big arrow and big node as shown in Figure 15.

Every black node is the child of the root node or a white node, so every L-tromino is involved in exactly one replacement. Also, the parent of any white node is a black node, whose parent, in turn, is a white node or the root. So the starting point of every big arrow will be on a big node. Therefore we obtain an L-tromino tiling forming a tree.

This shows for $n > 1$ that if an $n \times n$ board can be tiled by L-trominoes forming a tree, then n is even, and an $n/2 \times n/2$ board can also be tiled by L-trominoes forming a tree. Since a 1×1 board can trivially be tiled, we conclude that the only values of n for which an $n \times n$ board can be tiled are $n = 2^k$.

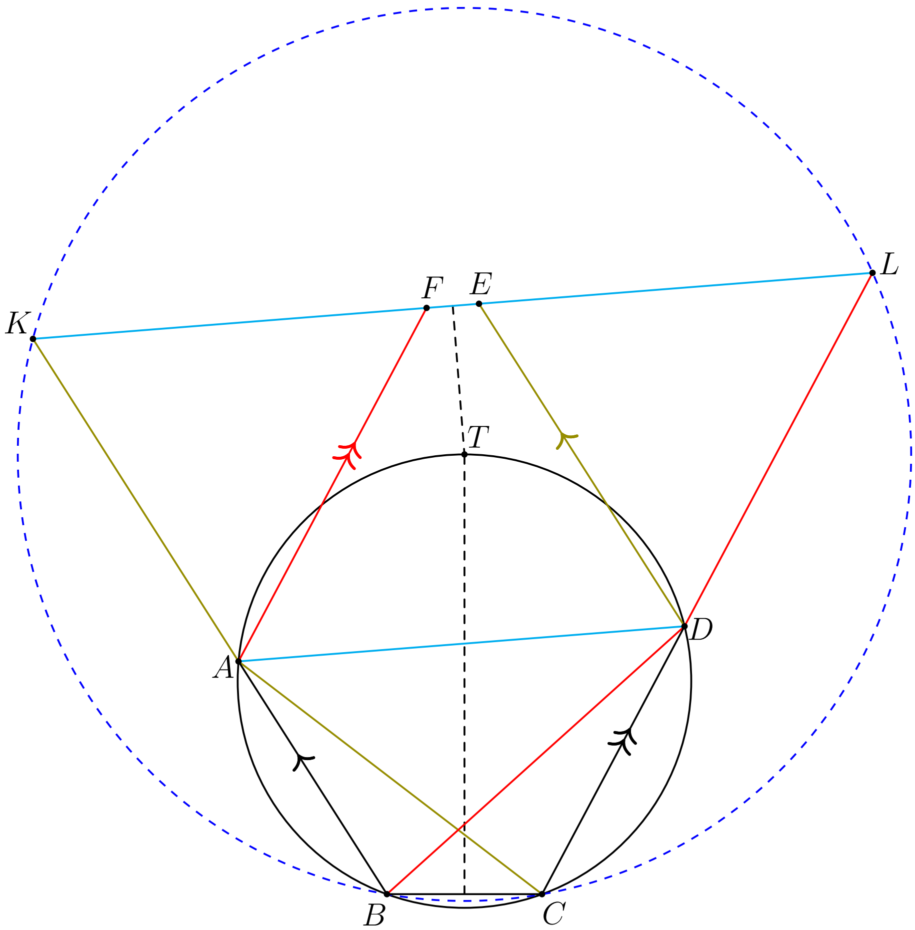
Geometry

G1. Let $ABCD$ be a cyclic quadrilateral such that $AC < BD < AD$ and $\angle DBA < 90°$. Point E lies on the line through D parallel to AB such that E and C lie on opposite sides of line AD, and $AC = DE$. Point F lies on the line through A parallel to CD such that F and C lie on opposite sides of line AD, and $BD = AF$.

Prove that the perpendicular bisectors of segments BC and EF intersect on the circumcircle of $ABCD$.

(Ukraine)

Solution 1. Let T be the midpoint of arc $\widehat{BAC}$ and let lines BA and CD intersect EF at K and L, respectively. Note that T lies on the perpendicular bisector of segment BC.



Since $ABCD$ is cyclic, $\frac{BD}{\sin \angle BAD} = \frac{AC}{\sin \angle ADC}$. From parallel lines we have $\angle DAF = \angle ADC$ and $\angle BAD = \angle EDA$. Hence,

$$AF \cdot \sin \angle DAF = BD \cdot \sin \angle ADC = AC \cdot \sin \angle BAD = DE \cdot \sin \angle EDA.$$

So F and E are equidistant from the line AD, meaning that EF is parallel to AD.

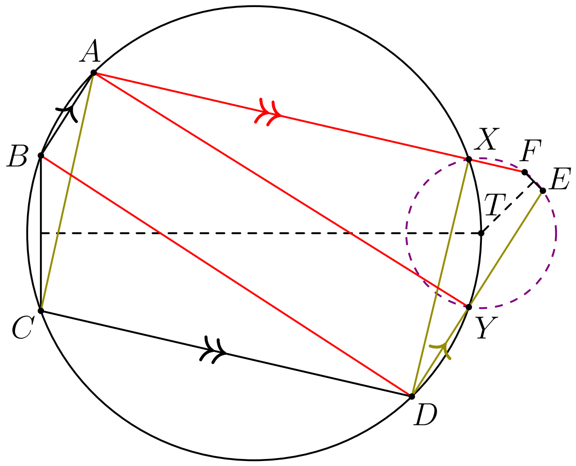
We have that $KADE$ and $FADL$ are parallelograms, hence we get $KA = DE = AC$ and $DL = AF = BD$. Also, $KE = AD = FL$ so it suffices to prove the perpendicular bisector of KL passes through T.

Triangle AKC is isosceles so $\angle BTC = \angle BAC = 2\angle BKC$. Likewise, $\angle BTC = 2\angle BLC$. Since T, K, and L all lie on the same side of BC and T lies on the perpendicular bisector of BC, T is the centre of circle $BKLC$. The result follows.

Solution 2. Let AF and DE meet ω at X and Y, respectively, and let T be as in Solution 1.

As $BD < AD$, $DY \parallel AB$ and $\angle BAY = \angle DBA < 90^\circ$, we have $DY < AB$ and Y lies on the opposite side of line AD to C. Also from $BD < AD$, we have B, C, and D all lie on the same side of the perpendicular bisector of AB which shows $AC > AB$. Combining these, we get $DY < AB < AC = DE$ and, as Y and E both lie on the same side of line AD, Y lies in the interior of segment DE. Similarly, X lies in the interior of segment DF.

Since AB is parallel to DY, we have $YA = BD = FA$. Likewise $XD = AC = ED$.



Claim 1. T is the midpoint of arc $\widehat{XY}$.

Proof. From $AX \parallel CD$ and $AB \parallel DY$ we have

$$\angle CAX = \angle AXD = \angle AYD = \angle YDB.$$

Since T is the midpoint of arc $\widehat{BAC}$, we have $\angle BAT = \angle TDC$, so

$$\angle TAX = \angle CAX + \angle BAC - \angle BAT = \angle YDB + \angle BDC - \angle TDC = \angle YDT. \qquad \square$$

Recall from above we have $AB < AC$ and analogously, $DC < DB$, which shows that X, Y and T all lie on the same side of line AD. In particular, T and A lie on opposite sides of XY so T lies on the internal angle bisector of $\angle XAY$. Since $AF = AY$, we have $\triangle ATF \cong \triangle ATY$, giving $TF = TY$.

Likewise, $TE = TX$, so $TE = TF$, meaning that T lies on the perpendicular bisector of segment EF as required.

Comment. The statement remains true without the length and angle conditions on cyclic quadrilateral $ABCD$ however additional care is required to consider different cases based on the ordering of points on lines DE and AF. It is also possible for T to be on the external angle bisector of $\angle XAY$.

Solution 3. From $AF = DB$, $AC = DE$ and

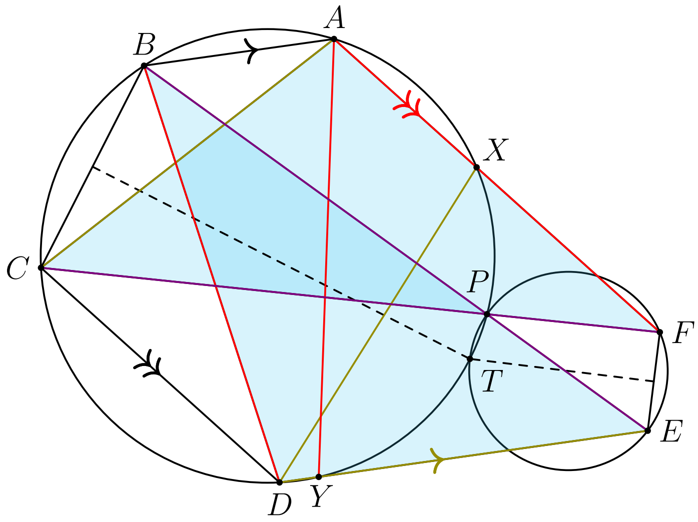
$$\angle(AC, AF) = \angle(AC, CD) = \angle(AB, BD) = \angle(DE, DB),$$

triangles ACF and DEB are congruent, so $CF = BE$.

Let $P = BE \cap CF$. Since

$$\angle(CP, BP) = \angle(CF, BE) = \angle(AF, DB) = \angle(DC, DB),$$

we have that P lies on circle $ABCD$.



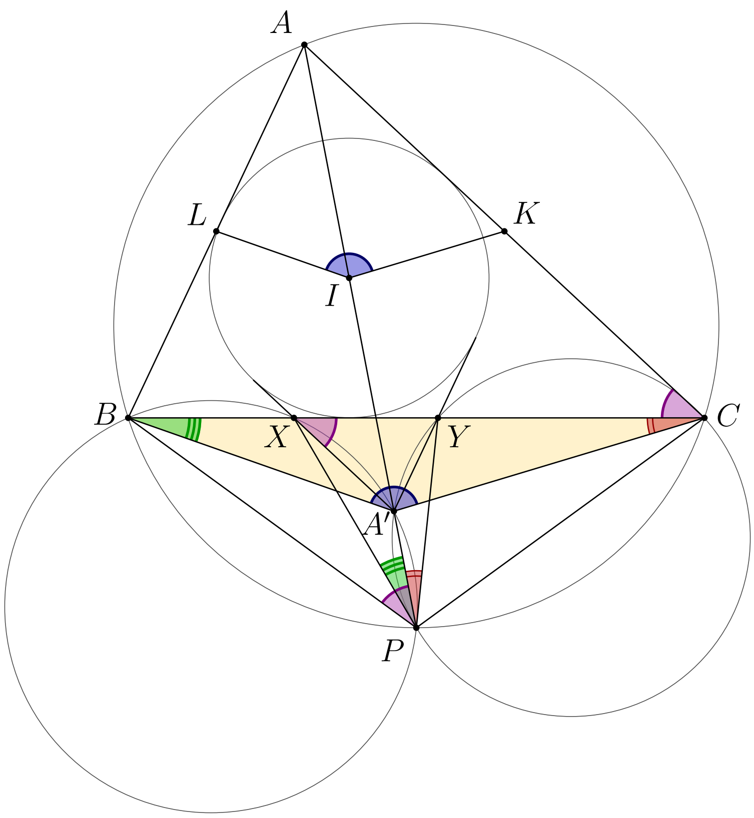
Finally, let T be the Miquel point of the quadrilateral $BCFE$ so T lies on circles EFP and $ABCD$. Note that T is the centre of spiral similarity taking segments BE to CF and since $BE = CF$, this is in fact just a rotation, so $TB = TC$ and $TE = TF$; that is, the perpendicular bisectors of BC and EF meet at T, on circle $ABCD$.

G2. Let ABC be a triangle with $AB < AC < BC$, incentre I and incircle ω. Let X be the point in the interior of side BC such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point in the interior of side BC such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AB and AC, respectively.

Prove that $\angle KIL + \angle YPX = 180^\circ$.

(Poland)

Solution 1. Let A' be the reflection of A in I, then A' lies on the angle bisector AP. Lines $A'X$ and $A'Y$ are the reflections of AC and AB in I, respectively, and so they are the tangents to ω from X and Y. As is well-known, $PB = PC = PI$, and since $\angle BAP = \angle PAC > 30^\circ$, $PB = PC$ is greater than the circumradius. Hence $PI > \frac{1}{2}AP > AI$; we conclude that A' lies in the interior of segment AP.

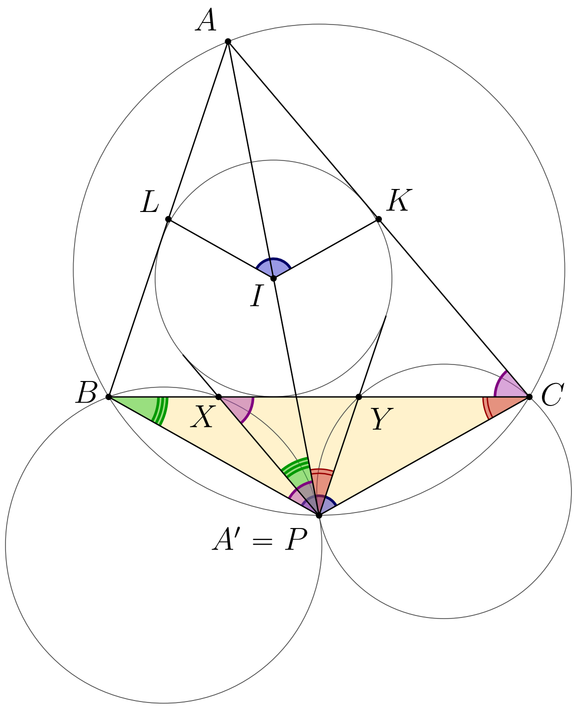


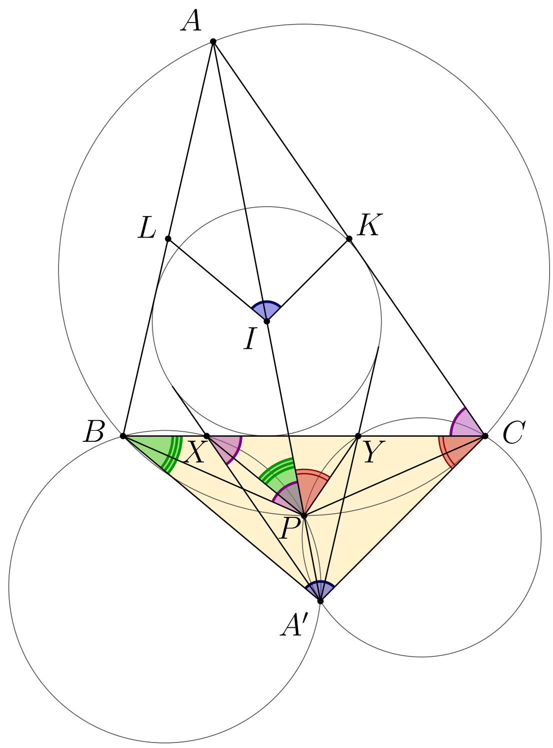
We have $\angle APB = \angle ACB$ in the circumcircle and $\angle ACB = \angle A'XC$ because $A'X \parallel AC$. Hence, $\angle APB = \angle A'XC$, and so quadrilateral $BPA'X$ is cyclic. Similarly, it follows that $CYA'P$ is cyclic.

Now we are ready to transform $\angle KIL + \angle YPX$ to the sum of angles in triangle $A'CB$. By a homothety of factor 2 at A we have $\angle KIL = \angle CA'B$. In circles $BPA'X$ and $CYA'P$ we have $\angle APX = \angle A'BC$ and $\angle YPA = \angle BCA'$, therefore

$$\angle KIL + \angle YPX = \angle CA'B + (\angle YPA + \angle APX) = \angle CA'B + \angle BCA' + \angle A'BC = 180^\circ.$$

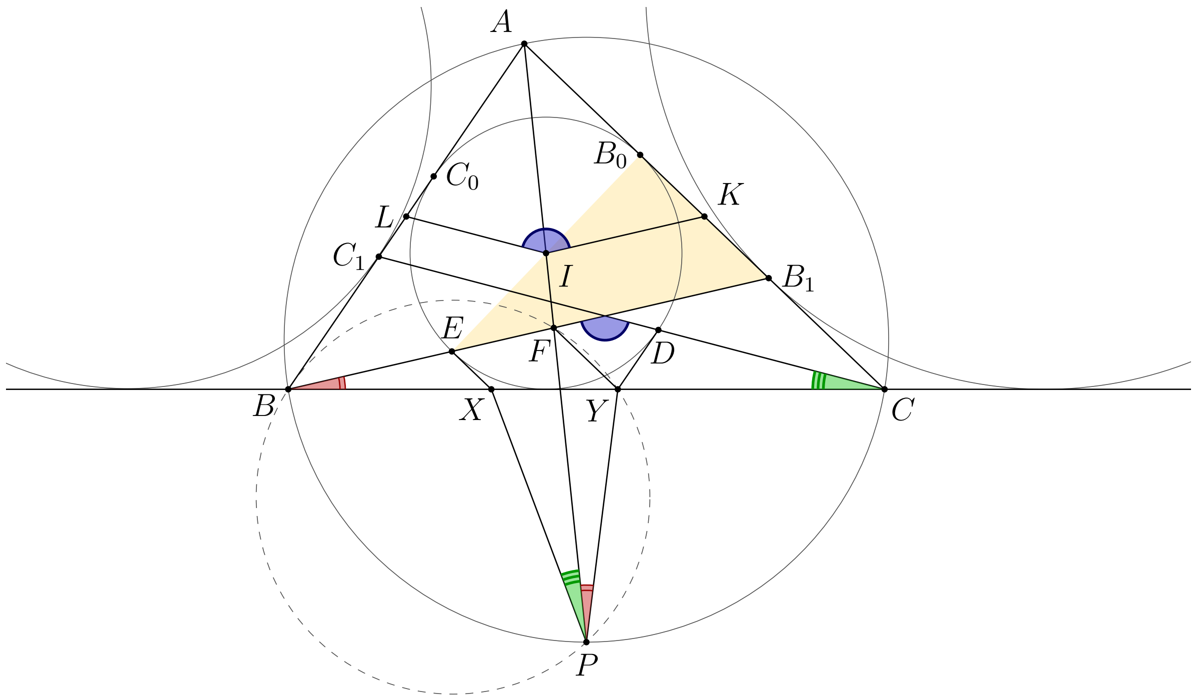
Comment. The constraint $AB < AC < BC$ was added by the Problem Selection Committee in order to reduce case-sensitivity. Without that, there would be two more possible configurations according to the possible orders of points A, P and A', as shown in the pictures below. The solution for these cases is broadly the same, but some extra care is required in the degenerate case when A' coincides with P and line AP is a common tangent to circles BPX and CPY.





Solution 2. Let $BC = a$, $AC = b$, $AB = c$ and $s = \frac{a+b+c}{2}$, and let the radii of the incircle, B-excircle and C-excircle be r, r_b and r_c, respectively. Let the incircle be tangent to AC and AB at B_0 and C_0, respectively; let the B-excircle be tangent to AC at B_1, and let the C-excircle be tangent to AB at C_1. As is well-known, $AB_1 = s - c$ and $\text{area}(\triangle ABC) = rs = r_c(s-c)$.

Let the line through X, parallel to AC be tangent to the incircle at E, and the line through Y, parallel to AB be tangent to the incircle at D. Finally, let AP meet BB_1 at F.



It is well-known that points B, E, and B_1 are collinear by the homothety between the incircle and the B-excircle, and $BE \parallel IK$ because IK is a midline in triangle B_0EB_1. Similarly, it follows that C, D, and C_1 are collinear and $CD \parallel IL$. Hence, the problem reduces to proving $\angle YPA = \angle CBE$ (and its symmetric counterpart $\angle APX = \angle DCB$ with respect to the vertex C), so it suffices to prove that $FYPB$ is cyclic. Since $ACPB$ is cyclic, that is equivalent to $FY \parallel B_1C$ and $\frac{BF}{FB_1} = \frac{BY}{YC}$.

By the angle bisector theorem we have

$$\frac{BF}{FB_1} = \frac{AB}{AB_1} = \frac{c}{s-c}.$$

The homothety at C that maps the incircle to the C-excircle sends Y to B, so

$$\frac{BC}{YC} = \frac{r_c}{r} = \frac{s}{s-c}.$$

So,

$$\frac{BY}{YC} = \frac{BC}{YC} - 1 = \frac{s}{s-c} - 1 = \frac{c}{s-c} = \frac{BF}{FB_1},$$

which completes the solution.

G3. Let $ABCDE$ be a convex pentagon and let M be the midpoint of AB. Suppose that segment AB is tangent to the circumcircle of triangle CME at M and that D lies on the circumcircles of triangles AME and BMC. Lines AD and ME intersect at K, and lines BD and MC intersect at L. Points P and Q lie on line EC so that $\angle PDC = \angle EDQ = \angle ADB$.

Prove that lines KP, LQ, and MD are concurrent.

(Belarus)

Common remarks. Each of solutions we present consists of three separate parts:

(a) proving $KP \parallel MC$ and $LQ \parallel ME$;

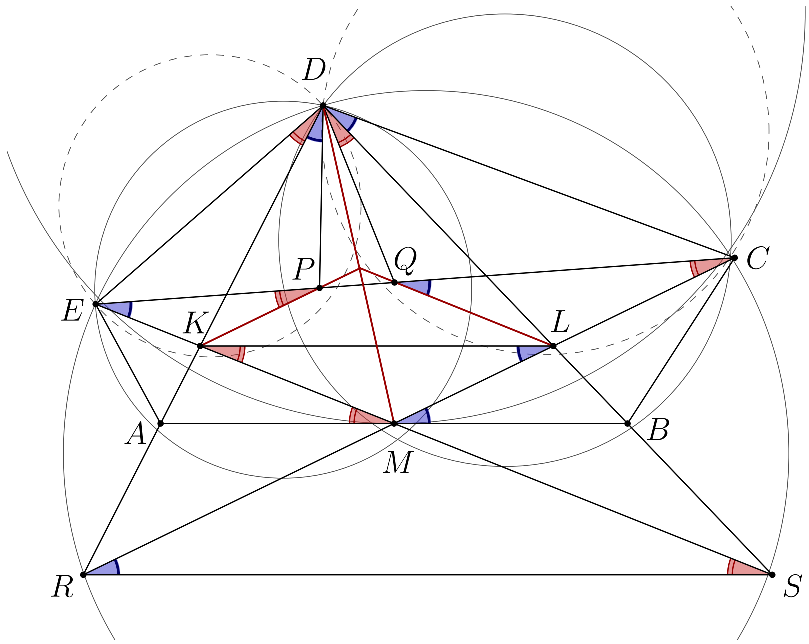
(b) proving $KL \parallel AB$ and, optionally, showing that points C, E, K, and L are concyclic;

(c) completing the solution either using homotheties or the parallelogram enclosed by lines KP, MK, ML and LQ, or radical axes between three circles.

Solution 1.

(a) Notice that the condition $\angle PDC = \angle ADB$ is equivalent to $\angle ADP = \angle BDC$, and $\angle EDQ = \angle ADB$ is equivalent to $\angle EDA = \angle QDB$. From line AB being tangent to circle CME, and circles $AMDE$ and $CDME$ we read $\angle ECM = \angle EMA = \angle EDA = \angle QDB$ and $\angle MEC = \angle BMC = \angle BDC = \angle ADP$.

Using $\angle ADP = \angle MEC$, the points D, E, K, and P are concyclic, which gives that $\angle EPK = \angle EDA = \angle ECM$. From that, we can see that $KP \parallel MC$. It can be shown similarly that C, D, Q, and L are concyclic, $\angle LQC = \angle MEC$ and therefore $LQ \parallel ME$.



(b) Let rays DA and DB intersect circle CDE at R and S, respectively. We now observe that $\angle SEC = \angle SDC = \angle MEC$, so points E, M, and S are collinear. We similarly obtain that C, M, and R are collinear.

From $\angle SRC = \angle SEC = \angle BMC$ we can see that $RS \parallel AB$. Since M bisects AB, it follows that $KL \parallel RS$.

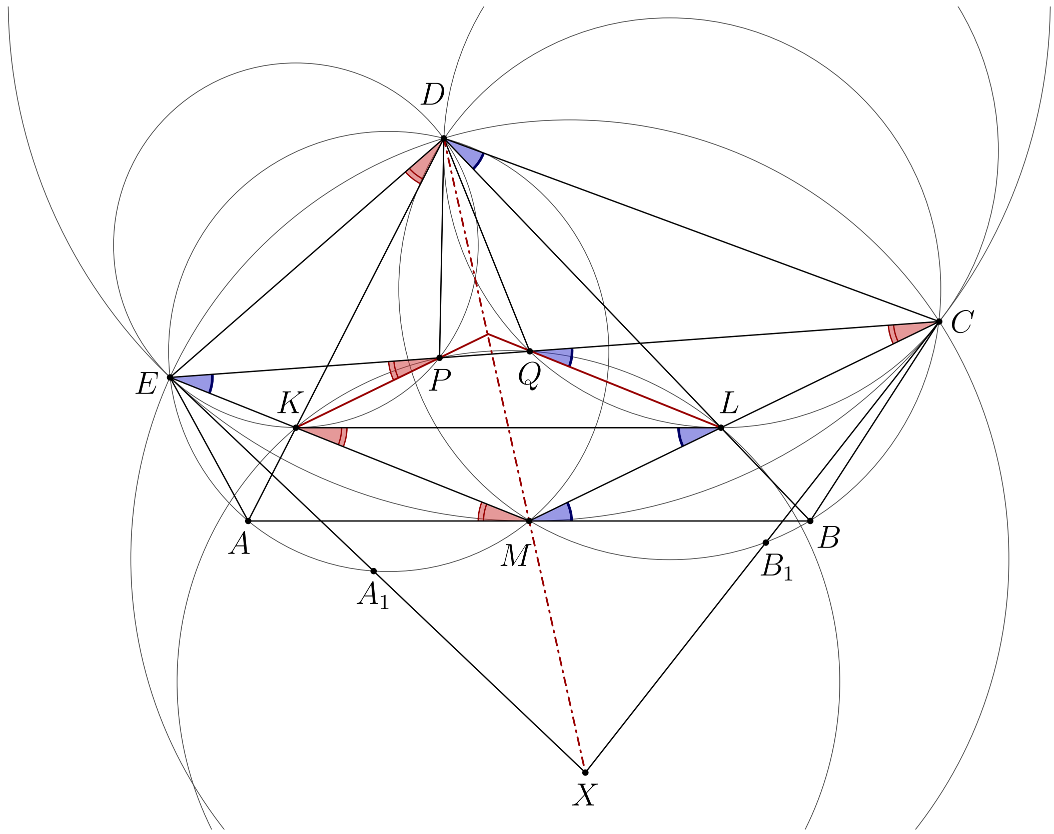
(c) Consider the homothety at D that sends RS to KL. Because $KP \parallel RC$ and $LQ \parallel SE$, that homothety sends the concurrent lines DM, RC, and SE to DM, KP, and LQ, so these lines are also concurrent.

Solution 2.

(a) As in Solution 1, we show the following: $\angle ECM = \angle EMA = \angle EDA = \angle EPK$; $\angle MEC = \angle BMC = \angle BDC = \angle LQC$; the points C, D, Q, and L are concyclic; the points D, E, K, and P are concyclic; $KP \parallel MC$; and $LQ \parallel ME$.

(b) Notice that triangles EKP and EMC are homothetic at E, so their circumcircles CME and $DEKP$ are tangent to each other at E. Similarly, circle $CDQL$ is tangent to circle CME at C.

Suppose that the tangents to circle CME at C and E intersect at point X. (The case when CE is a diameter in circle CME can be considered as a limit case.) Moreover, let EX and CX intersect circles $DEAM$ and $BCDM$ again at $A_1 \neq E$ and $B_1 \neq C$, respectively.



We have $XE = XC$ because they are the tangents from X to circle CME. Moreover, in circle $DEAM$, chords AM and A_1E are tangent to circle CME, so $A_1E = AM$. Similarly, we have $B_1C = BM$, hence $A_1E = AM = BM = B_1C$. We conclude $XA_1 = XB_1$, so the powers of X with respect to circles $DEAM$ and $BCDM$ are equal. Therefore, X lies on the radical axis of these two circles, which is DM.

Now notice that by $XC = XE$, point X has equal powers to circles $CDQL$ and $DEKP$, so DX is the radical axis of these circles. Point M lies on DX, so $ME \cdot MK = MC \cdot ML$; we conclude that C, E, K, and L are concyclic. Hence, by $\angle MKL = \angle ECM = \angle KMA$ we have $KL \parallel AB$.

(c) As $\angle EPK = \angle EMA = \angle QLK$, we have that $KLQP$ is cyclic. The radical axes between circles $DEKP$, $CDQL$ and $KLQP$ are DM, KP and LQ, so they are concurrent at the radical centre of the three circles.

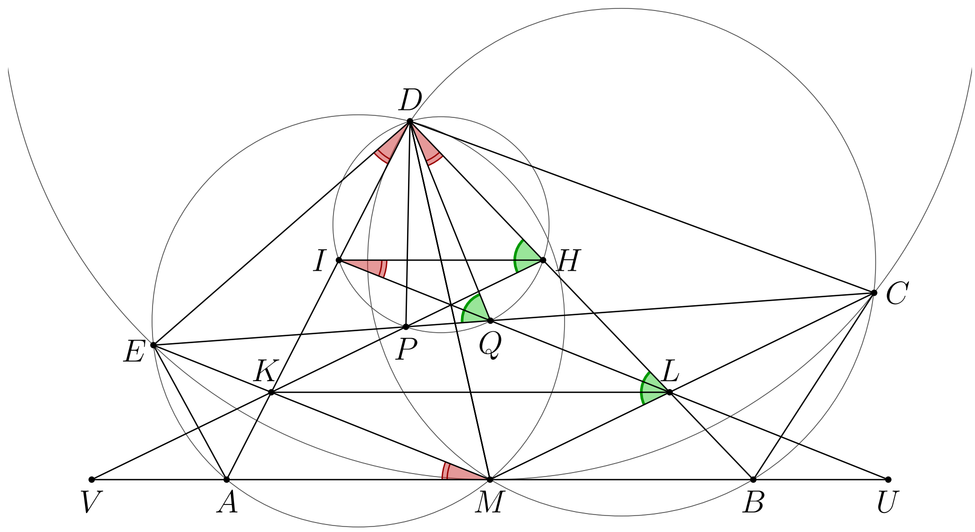
Solution 3.

(b) We present another proof that $KL \parallel AB$.

Let $AD \cap LQ = I$, $BD \cap KP = H$, $AB \cap LQ = U$ and $AB \cap KP = V$. Since

$$\angle DHP = \angle DLM = 180^\circ - \angle CLD = 180^\circ - \angle CQD = \angle DQE,$$

point H lies on circle DPQ. Similarly, we obtain that point I lies on this circle. Hence, $\angle LIH = \angle QDB = \angle EDA = \angle EMA$, and $LQ \parallel ME$ implies that $HI \parallel AB$.



Let $AM = BM = d$, then we have

$$\frac{BU}{IH} = \frac{BL}{LH} = \frac{BM}{MV} = \frac{d}{d+AV} \quad \text{and} \quad \frac{AV}{IH} = \frac{AK}{KI} = \frac{AM}{MU} = \frac{d}{d+BU}.$$

Hence, $BU \cdot (d + AV) = AV \cdot (d + BU)$, so $BU = AV$. Therefore, $\triangle MLU \cong \triangle VKM$ which implies $KL \parallel AB \parallel HI$.

(c) Lines MK, ML, KP and LQ enclose a parallelogram. Line DM passes through the midpoint of KL, which the centre of the parallelogram, and passes through the vertex M. Therefore, DM passes through the opposite vertex, which is the intersection of KP and LQ.

G4. Let $ABCD$ be a quadrilateral with AB parallel to CD and $AB < CD$. Lines AD and BC intersect at a point P. Point $X \neq C$ on the circumcircle of triangle ABC is such that $PC = PX$. Point $Y \neq D$ on the circumcircle of triangle ABD is such that $PD = PY$. Lines AX and BY intersect at Q.

Prove that PQ is parallel to AB.

(Ukraine)

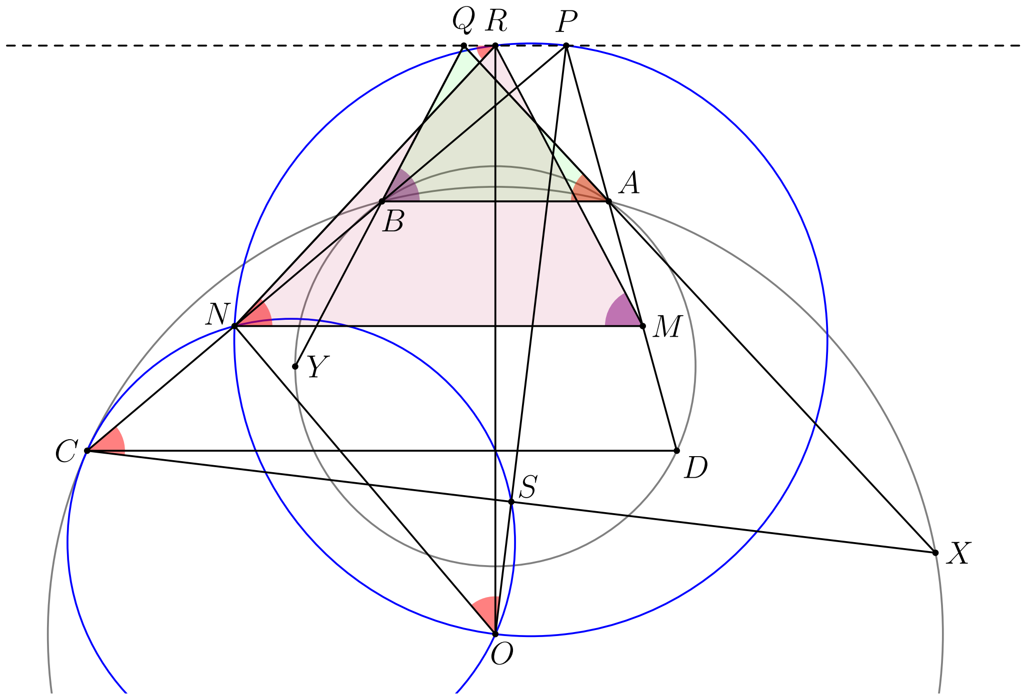
Solution 1. Let M and N be the midpoints of AD and BC, respectively and let the perpendicular bisector of AB intersect the line through P parallel to AB at R.

Lemma. Triangles QAB and RNM are similar.

Proof. Let O be the circumcentre of triangle ABC, and let S be the midpoint of CX. Since N, S, and R are the respective perpendicular feet from O to BC, CX, and PR, we have that quadrilaterals $PRNO$ and $CNSO$ are cyclic. Furthermore, P, S, and O are collinear as $PC = PX$. Since $ABCX$ is also cyclic, we have that

$$\angle QAB = \angle XCB = \angle PON = 180^\circ - \angle NRP = \angle MNR.$$

Analogously, we have that $\angle ABQ = \angle RMN$, so triangles QAB and RNM are similar. □

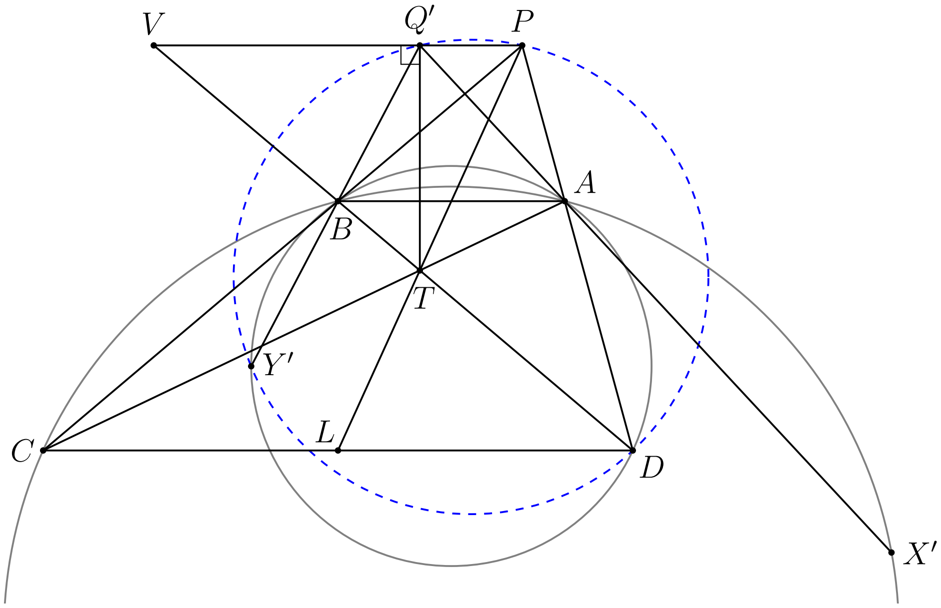


Let $d(Z, \ell)$ denote the perpendicular distance from the point Z to the line ℓ. Using that $PR \parallel AB$ along with the similarities $QAB \sim RNM$ and $PAB \sim PMN$, we have that

$$\frac{d(Q, AB)}{AB} = \frac{d(R, MN)}{MN} = \frac{d(P, MN)}{MN} = \frac{d(P, AB)}{AB},$$

which implies that $PQ \parallel AB$.

Solution 2. Let BD and AC intersect at T and let the line through P parallel to AB intersect BD at V. Next, let Q' be the foot of the perpendicular from T to PV. Finally, let $Q'A$ intersect circle ABC again at X' and $Q'B$ intersect circle ABD again at Y'.



Claim. PQ' bisects $\angle BQ'D$ externally.

Proof. Let PT intersect CD at L. Let ∞_{CD} be the point at infinity on line CD. From the standard Ceva-Menelaus configuration we have $(D, C; L, \infty_{CD})$ is harmonic. Hence projecting through P we have

$$-1 = (D, C; L, \infty_{CD}) = (D, B; T, V).$$

As $(D, B; T, V)$ is harmonic, and also $\angle VQ'T = 90^\circ$ (by construction), the claim follows. □

Now as

$$\angle Q'PD = \angle BAD = 180^\circ - \angle DY'B = 180^\circ - \angle DY'Q'$$

we have $Q'PDY'$ cyclic. By the claim, we have that P is the midpoint of arc $\widehat{DQ'Y'}$, so $PD = PY'$.

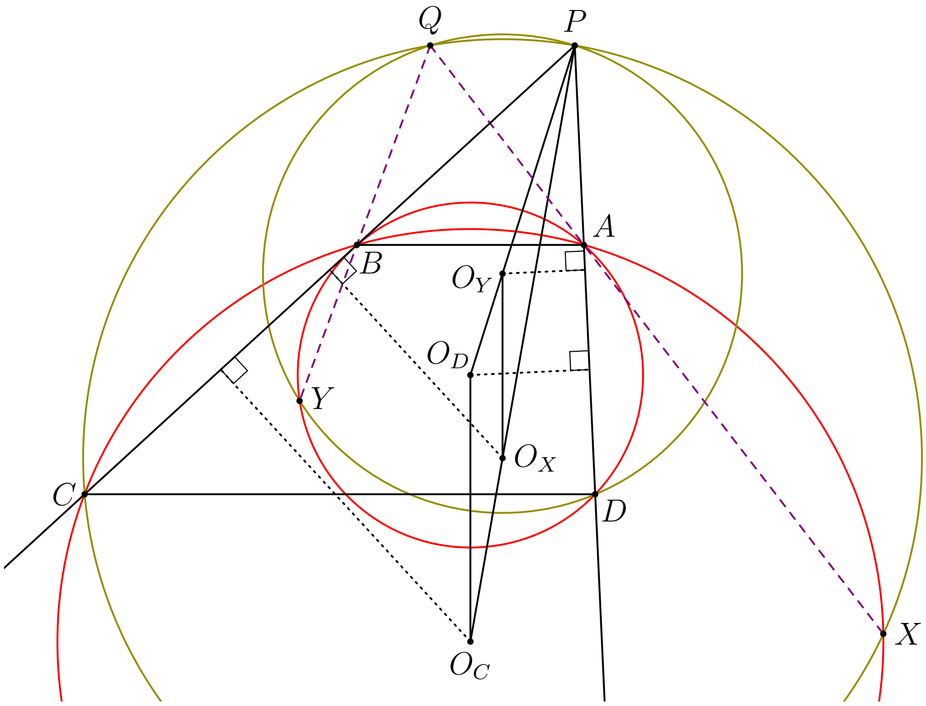
Since Y is the unique point not equal to D on circle ABD satisfying $PD = PY$, we have $Y' = Y$.

Likewise $X' = X$ so $Q' = Q$ and we are done.

Solution 3. Let AX intersect circle PCX for the second time at Q'. Then

$$\angle AQ'P = \angle XQ'P = \angle XCP = \angle XCB = 180^\circ - \angle BAX = \angle Q'AB$$

so PQ' is parallel to AB. Hence, it suffices to show that Q' is equal to Q. To do so, we aim to show the common chord of circles PCX and PDY is parallel to AB, since then by symmetry Q' is also the second intersection of BY and circle PDY.



Let the centres of circles PCX and PDY be O_X and O_Y, respectively. Let the centres of circles ABC and ABD be O_C and O_D, respectively.

Note P, O_X, and O_C are collinear since they all lie on the perpendicular bisector of CX. Likewise P, O_Y, and O_D are collinear on the perpendicular bisector of DY. By considering the projections of O_X and O_C onto BC, and O_Y and O_D onto AD, we have

$$\frac{PO_X}{PO_C} = \frac{\frac{PC}{2}}{\frac{PB+PC}{2}} = \frac{\frac{PD}{2}}{\frac{PA+PD}{2}} = \frac{PO_Y}{PO_D}.$$

Hence O_XO_Y is parallel to O_CO_D, which is perpendicular to AB as desired.

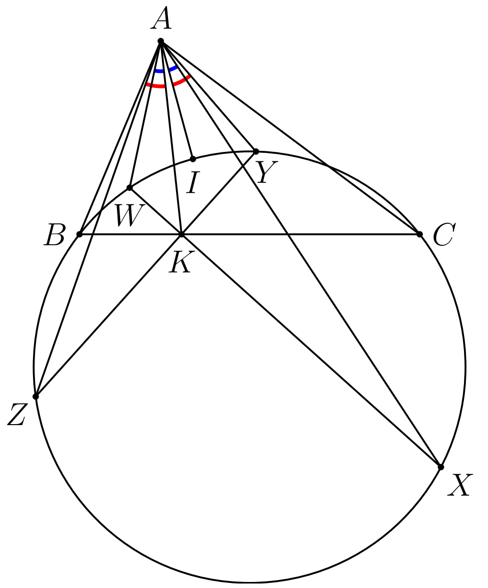
G5. Let ABC be a triangle with incentre I, and let Ω be the circumcircle of triangle BIC. Let K be a point in the interior of segment BC such that $\angle BAK < \angle KAC$. The angle bisector of $\angle BKA$ intersects Ω at points W and X such that A and W lie on the same side of BC, and the angle bisector of $\angle CKA$ intersects Ω at points Y and Z such that A and Y lie on the same side of BC.

Prove that $\angle WAY = \angle ZAX$.

(Uzbekistan)

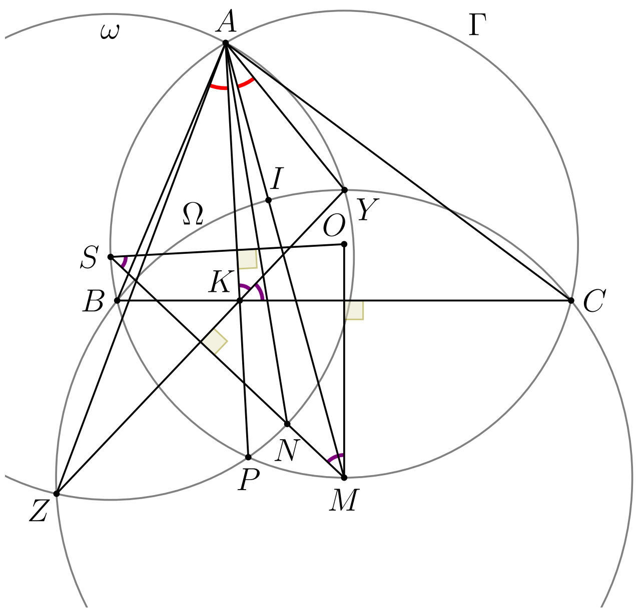
Common remarks. The key step in each solution is to prove that $\angle ZAK = \angle IAY$ and $\angle WAK = \angle IAX$. The problem is implied by these equalities, as we then have that

$$\angle WAY = \angle WAK + \angle KAI + \angle IAY = \angle IAX + \angle KAI + \angle ZAK = \angle ZAX.$$



We now present several proofs that $\angle ZAK = \angle IAY$, with $\angle WAK = \angle IAX$ following in an analogous manner.

Solution 1. Let Γ be circle ABC and ω be circle AYZ. Let O, M, and S be the centres of Γ, Ω, and ω, respectively. Let AK intersect Γ again at P, and let the angle bisector of $\angle ZAY$ intersect ω again at N.

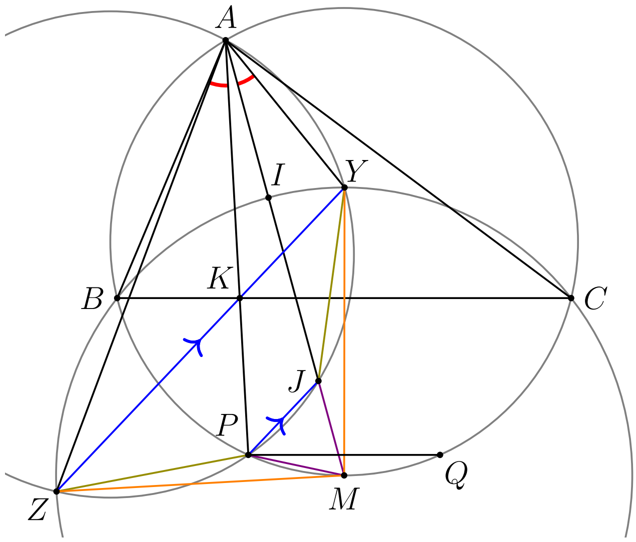


By power of a point from K to Γ and Ω, we have that $KA \cdot KP = KB \cdot KC = KY \cdot KZ$, so P also lies on ω. The pairwise common chords of Γ, Ω, and ω are then $AP \perp OS$, $BC \perp OM$, and $YZ \perp MS$, so we have that $\angle OMS = \angle CKY = \angle YKA = \angle MSO$. As M lies on Γ and $OM = OS$, S also lies on Γ. Note that N lies on MS as $NY = NZ$, so

$$\angle PAN = \frac{1}{2}\angle PSN = \frac{1}{2}\angle PSM = \frac{1}{2}\angle PAM.$$

Thus, AN bisects $\angle PAM$ in addition to $\angle ZAY$, which means that $\angle ZAK = \angle IAY$ as K lies on AP and I lies on AM.

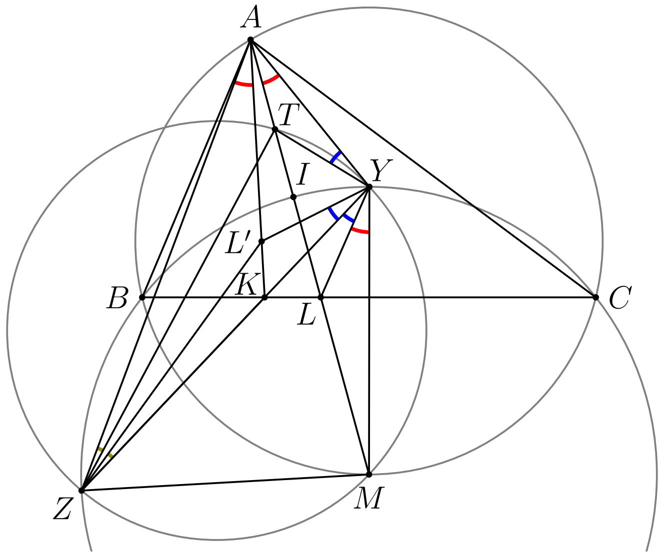
Solution 2. Define M and P as in Solution 1, and recall that $AYPZ$ is cyclic. Let Q be the second intersection of the line parallel to BC through P with circle ABC and let J be the incentre of triangle APQ.



Since PQ is parallel to BC and $\angle BAP < \angle PAC$, the angle bisector of $\angle APQ$ is parallel to the angle bisector of $\angle AKC$. Hence, PJ is parallel to YZ. As M is the midpoint of $\widehat{PQ}$ on circle APQ, we have that $MP = MJ$. Then since segments YZ and PJ are parallel and have a common point M on their perpendicular bisectors, $PJYZ$ is cyclic with $JY = PZ$. It follows that J also lies on circle $AYPZ$ and that $\angle ZAP = \angle JAY = \angle IAY$.

Comment. The proof of the analogous case of $\angle WAK = \angle IAX$ is slightly different. In this case, J should be defined as the A-excentre of APQ so that PJ is the external bisector of $\angle APQ$ and $PJ \parallel WX$. The proof is otherwise exactly the same.

Solution 3. As in the previous solutions, let M be the centre of Ω. Let L be the intersection of AM and BC, and let L' be the reflection of L over YZ. Let the circle MYZ intersect AM again at T.



Note that as M is the midpoint of $\widehat{BC}$ on circle ABC and L is the foot of the bisector of $\angle BAC$, we have that $MA \cdot ML = MI^2 = MY^2$. It follows by power of a point that MY is tangent to circle ALY, so $\angle LAY = \angle LYM$. Using directed angles, we then have that

$$\measuredangle AYT = \measuredangle MTY - \measuredangle MAY = \measuredangle MZY - \measuredangle LYM = \measuredangle ZYM - \measuredangle LYM = \measuredangle ZYL = \measuredangle L'YZ,$$

where we use the fact that $MY = MZ$ and that L and L' are symmetric about YZ. Thus, YT and YL' are isogonal in $\angle AYZ$. Analogously, we have that ZT and ZL' are isogonal in $\angle YZA$. This means that T and L' are isogonal conjugates in triangle AYZ, which allows us to conclude that $\angle ZAK = \angle IAY$ since L' lies on AK and T lies on AI.

Comment. Owing to the condition $\angle BAK < \angle KAC$, points L' and T lie inside triangle AYZ. However, if one tries to write down the same proof for $\angle WAK = \angle IAX$, the analogues L'_1 and T_1 of L' and T would lie outside triangle AWX. Thus, the solution has been written using directed angles so that it applies directly to this case as well. It is also possible that L'_1 lies on circle AWX and T_1 is a point at infinity. In this case, it is straightforward to interpret the directed angle chase to prove the isogonality, and the isogonality also follows from this scenario being a limit case of other configurations.

Note. The original proposal remarks that this problem is a special case of a more general property:

A convex quadrilateral $ABCD$ is inscribed in a circle ω. The bisectors between AC and BD intersect ω at four points, forming a convex quadrilateral $PQRS$. Then the conditions

$$XA \cdot XC = XB \cdot XD \qquad \text{and} \qquad \measuredangle(XP, XQ) = \measuredangle(XS, XR)$$

on point X are equivalent.

The Problem Selection Committee believes that the proof of this generalisation is beyond the scope of the competition and considers the original problem to be more suitable.

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G6. Let ABC be an acute triangle with $AB < AC$, and let Γ be the circumcircle of ABC. Points X and Y lie on Γ so that XY and BC intersect on the external angle bisector of $\angle BAC$. Suppose that the tangents to Γ at X and Y intersect at a point T on the same side of BC as A, and that TX and TY intersect BC at U and V, respectively. Let J be the centre of the excircle of triangle TUV opposite the vertex T.

Prove that AJ bisects $\angle BAC$.

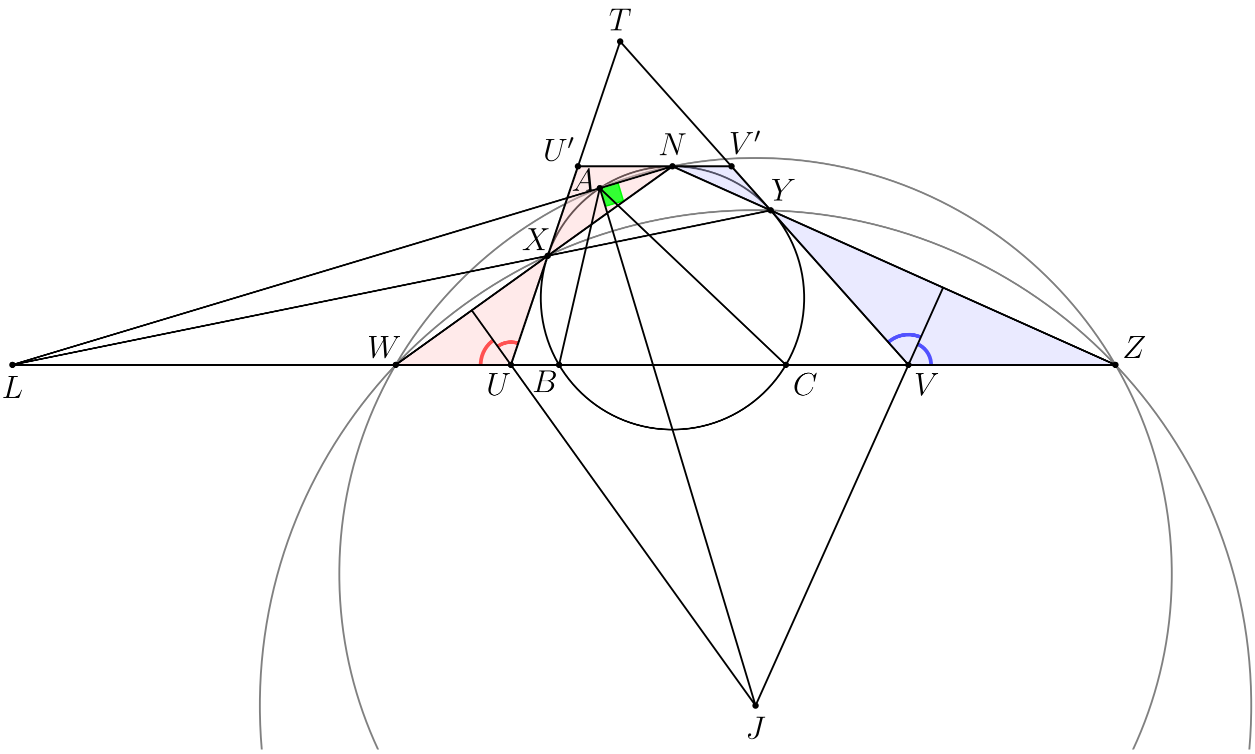
(Poland)

Solution 1. Let N be the midpoint of $\widehat{BAC}$ on Γ, and let NX and NY intersect BC at W and Z, respectively.

Claim. Quadrilateral $WXYZ$ is cyclic, and its circumcentre is J.

Proof. As N is the midpoint of $\widehat{BAC}$, W and Z lie on BC, and X and Y are the second intersections of NW and NZ with Γ, we have that $WXYZ$ is cyclic.

Let the parallel to BC through N intersect TU and TV at U' and V', respectively. Then U' is the intersection of the tangents to Γ at N and X, so $U'N = U'X$. As $NU' \parallel BC$, $U'NX$ is similar to UWX, so $UW = UX$ as well. Hence, the perpendicular bisector of WX is the internal bisector of $\angle XUW$, which is the external bisector of $\angle VUT$. Analogously, the perpendicular bisector of YZ is the external bisector of $\angle TVU$. This means that the circumcentre of $WXYZ$ is the intersection of the external bisectors of $\angle VUT$ and $\angle TVU$, which is J. $\square$



Let AN intersect BC at L, so XY passes through L as well. By power of a point from L to Γ and circle $WXYZ$, we have that $LA \cdot LN = LX \cdot LY = LW \cdot LZ$, so $WANZ$ is also cyclic. Thus, A is the Miquel point of quadrilateral $WXYZ$. As $WXYZ$ is cyclic with circumcentre J and its opposite sides WX and YZ intersect at N, we have that $AN \perp AJ$. Since AN is the external bisector of $\angle BAC$, this implies that AJ is the internal bisector of $\angle BAC$.

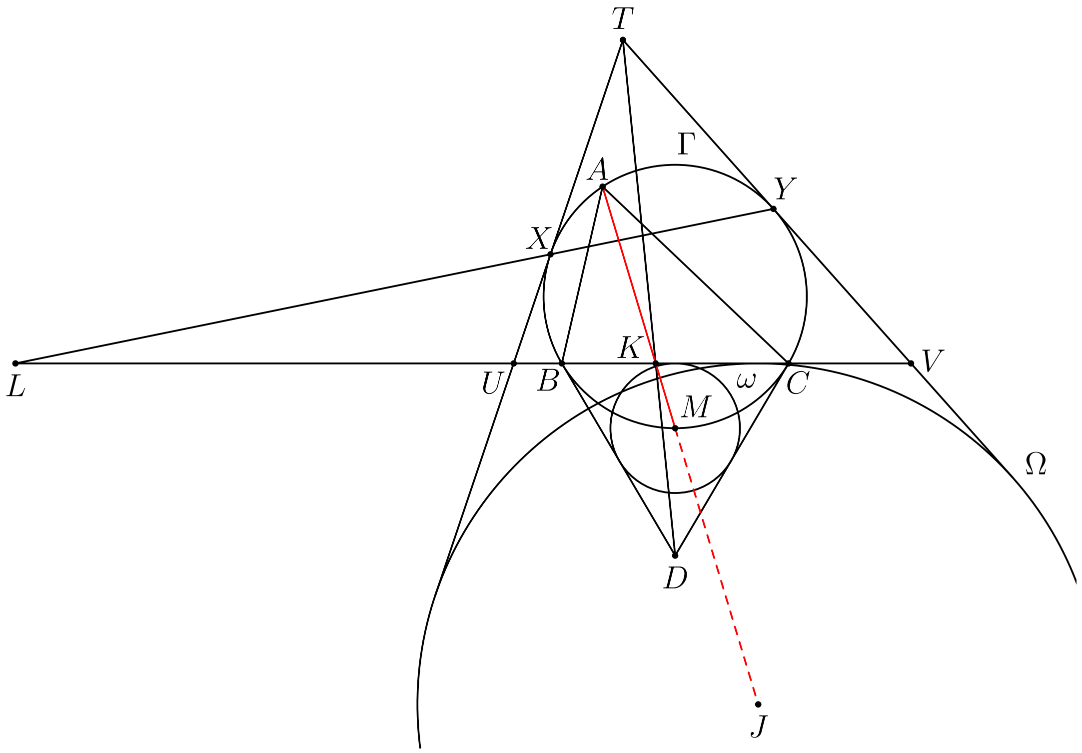
Solution 2. Let the internal and external angle bisectors of $\angle BAC$ intersect BC at K and L, respectively. Let AK intersect circle ABC again at M, and let D be the intersection of the tangents to Γ at B and C. Let Ω be the T-excircle of TUV, and let ω be the incircle of DBC.

Claim. The points T, K, and D are collinear.

Proof. Note that BC and XY are the polars of T and D with respect to Γ. By La Hire's Theorem, TD is the polar of L with respect to Γ. As $(B,C;K,L)=-1$, K also lies on the polar of L, thus proving the collinearity. □

Claim. The incentre of DBC is M.

Proof. We have that $\angle MBC = \angle MAC = \frac{1}{2}\angle BAC = \frac{1}{2}\angle DBC$, so BM bisects $\angle DBC$. Similarly, CM bisects $\angle BCD$, so M is the incentre of DBC. □



Claim. The intersection of the common external tangents of Ω and ω is K.

Proof. Let K' be the intersection of the common external tangents of Ω and ω. As Ω and ω are both tangent to BC and lie on the same side of BC opposite to A, K' lies on BC. As T is the intersection of the common external tangents of Γ and Ω and D is the intersection of the common external tangents of Γ and ω, by Monge's theorem K' lies on TD. As K' lies on both BC and TD, it is the same point as K. □

Hence, K is collinear with the centres of Ω and ω, which are M and J, respectively. As K and M both lie on the bisector of $\angle BAC$, so does J.

Note. It can be shown that circles AUV and ABC are tangent and that the tangents from U and V to circle ABC different from TU and TV intersect at a point W on line TK. Reframing the problem in terms of quadrilateral $TUWV$ using these properties, we obtain the following problem:

Let $ABCD$ be a convex quadrilateral with an incircle ω, and let AC and BD intersect at P. Point E lies on ω such that the circumcircle of ACE is tangent to ω. Prove that if B and E lie on the same side of line AC, then the centre of the excircle of triangle ABC opposite the vertex B lies on line EP.

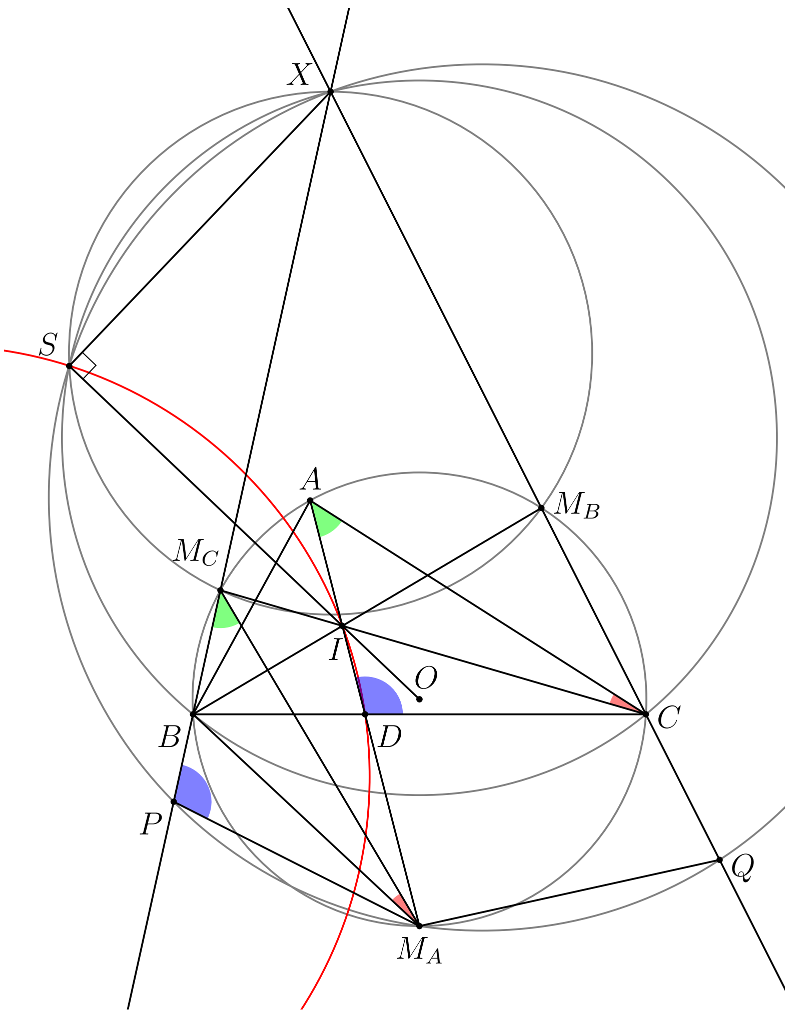
While this is an appealing statement, the Problem Selection Committee is uncertain about its difficulty and whether it has solutions that do not proceed by reducing to the original problem. Thus, it is believed that the original statement is more suitable for the competition.

G7. Let ABC be a triangle with incentre I such that $AB < AC < BC$. The second intersections of AI, BI, and CI with the circumcircle of triangle ABC are M_A, M_B, and M_C, respectively. Lines AI and BC intersect at D and lines BM_C and CM_B intersect at X. Suppose the circumcircles of triangles XM_BM_C and XBC intersect again at $S \neq X$. Lines BX and CX intersect the circumcircle of triangle SXM_A again at $P \neq X$ and $Q \neq X$, respectively.

Prove that the circumcentre of triangle SID lies on PQ.

(Thailand)

Solution 1.



Let O be the circumcentre of $\triangle ABC$. First we note from standard properties of the Miquel point S we have:

- $\triangle SM_CM_B \sim \triangle SBC \sim \triangle SPQ$; $(*)$
- I and S are inverses with respect to circle ABC;
- $\angle OSX = 90^\circ$.

Claim 1. $\angle M_APB = \angle CDA$.

Proof. From the above we have $\triangle OM_AI \sim \triangle OSM_A$ and

$$\angle M_APB = \angle M_APX = \angle M_ASX = 90^\circ + \angle M_ASO = 90^\circ + \angle OM_AI = \angle M_ABA = \angle CDA. \quad \square$$

Claim 2. $\frac{M_CB}{BP} = \frac{M_BC}{CQ} = \frac{AI}{ID}$.

Proof. Observe that $\angle PM_CM_A = \angle BM_CM_A = \angle DAC$ and $\angle M_CM_AB = \angle ICD$. Combining these with Claim 1 shows $M_CPM_AB \sim ADCI$. Therefore, $\frac{M_CB}{BP} = \frac{AI}{ID}$. Similarly, $\frac{M_BC}{CQ} = \frac{AI}{ID}$. □

Claim 3. $\frac{DP}{DQ} = \frac{IB}{IC}$.

Proof. Firstly, observe that $\angle ICB = \angle AM_BM_C$ and $\angle CBI = \angle M_BM_CA$ which gives that $\triangle IBC \sim \triangle AM_CM_B$. This, combined with Claim 2, is enough to show $\triangle DPQ \sim \triangle IBC$ by linearity, proving the claim. □

Claim 4. $\frac{IP}{IQ} = \frac{IB}{IC}$.

Proof. Combining $\triangle IBM_C \sim \triangle ICM_B$ with Claim 2 shows $IBM_CP \sim ICM_BQ$ giving the result. □

Finally, we have that

$$\frac{SP}{SQ} = \frac{SB}{SC} = \frac{BM_C}{CM_B} = \frac{IB}{IC}$$

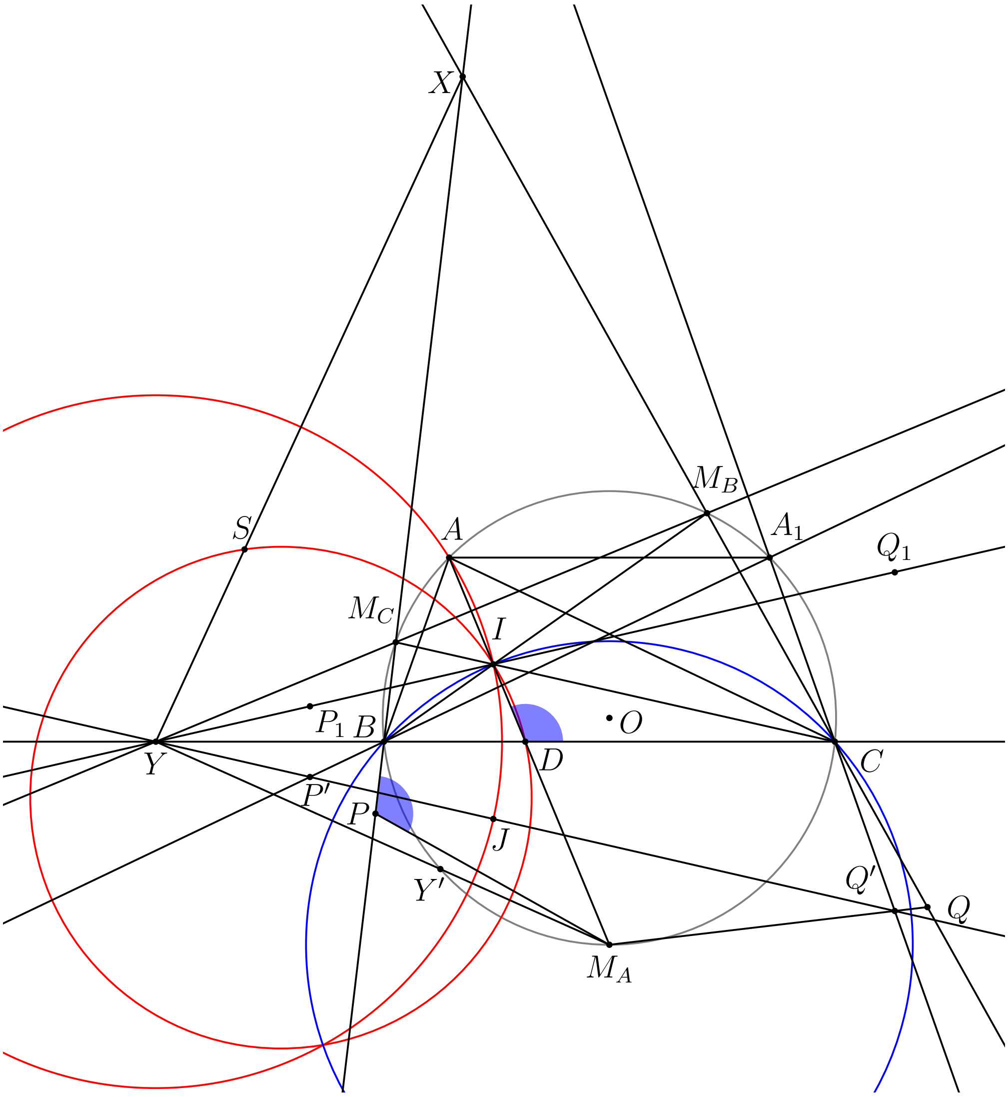
from $(*)$ and $\triangle IBM_C \sim \triangle ICM_B$. Putting this together with Claims 3 and 4, we have that

$$\frac{IB}{IC} = \frac{DP}{DQ} = \frac{IP}{IQ} = \frac{SP}{SQ},$$

which shows that circle SID is an Apollonius circle with respect to P and Q, giving the desired conclusion.

Comment. The condition $AB < AC$ ensures $S \neq X$. We also need to avoid the case $\angle BAC = 60^\circ$ as then $BM_C \parallel CM_B$.

Solution 2. We use Claim 1 from Solution 1. We will show that P and Q are inverses in circle SID which implies the result. Perform an inversion in circle BIC and denote the inverse of a point $\bullet$ by $\bullet'$.



Claim 1. $S' = J$ where J is the reflection of I across BC.

Proof. We have that S and I are inverses in circle ABC. Inverting this assertion in circle BIC shows that S' and I are inverses with respect to line BC, which is just a reflection in line BC. □

Let $Y = M_BM_C \cap BC$. From $\angle IM_CM_B = \angle M_BM_CA$ and $\angle AM_BM_C = \angle M_CM_BI$, we see that A and I are reflections in line M_BM_C so $YA = YI$. We have that circle SID maps to circle AIJ which, from the previous comment, has centre Y. Inverting the conclusion that P and Q are inverses with respect to circle SID in circle BIC, it suffices to show P' and Q' are inverses with respect to circle AIJ or equivalently, that $YP' \cdot YQ' = YA^2$.

Claim 2. Circle XSM_A maps to line YJ under the inversion in circle BIC.

Proof. Since circle BIC has centre M_A, the inverse of this circle is a line. By Claim 1, this line passes through J hence it suffices to prove that circle XSM_A passes through Y'. From inverting line BC in circle BIC, we have that BCM_AY' is cyclic so

$$YS \cdot YX = YB \cdot YC = YY' \cdot YM_A.$$

where we have used that Y, S and X are collinear by a standard property of the Miquel point. Hence Y' lies on circle XSM_A as required. □

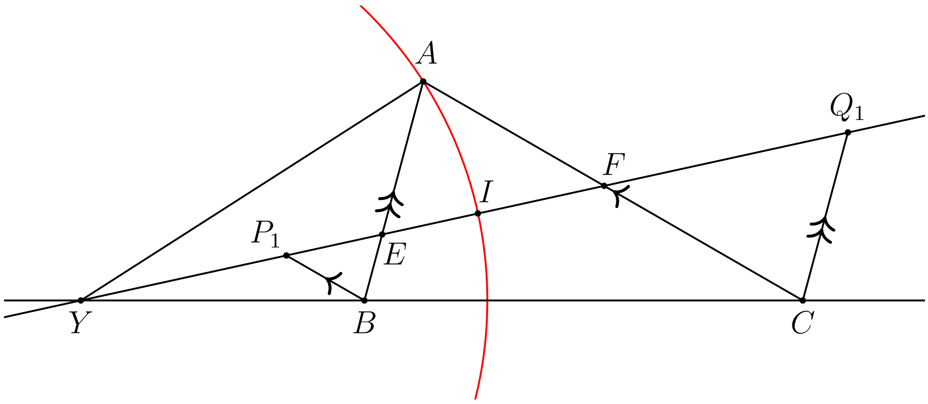
Let A_1 be the reflection of A in the perpendicular bisector of BC. Using Claim 1 from Solution 1,

$$\angle P'BM_A = \angle M_APB = \angle CDA = 180^\circ - \angle ACM_A = 180^\circ - \angle M_ABA_1.$$

Hence, P', B, and A_1 are collinear. Similarly Q', C, and A_1 are collinear. Let P_1 and Q_1 be the reflections of P' and Q' across BC. As P' and Q' lie on line YJ, it follows that P_1 and Q_1 lie on line YI. Also from the previous collinearities, we get $BP_1 \parallel AC$ and $CQ_1 \parallel AB$.

We have now reduced the problem to the following:

Claim 3 (Inverted Problem). Let ABC be a triangle with incentre I. Let Y be the point on BC such that $YA = YI$. Let P_1 and Q_1 be points on YI such that $BP_1 \parallel AC$ and $CQ_1 \parallel AB$. Then $YA^2 = YP_1 \cdot YQ_1$.

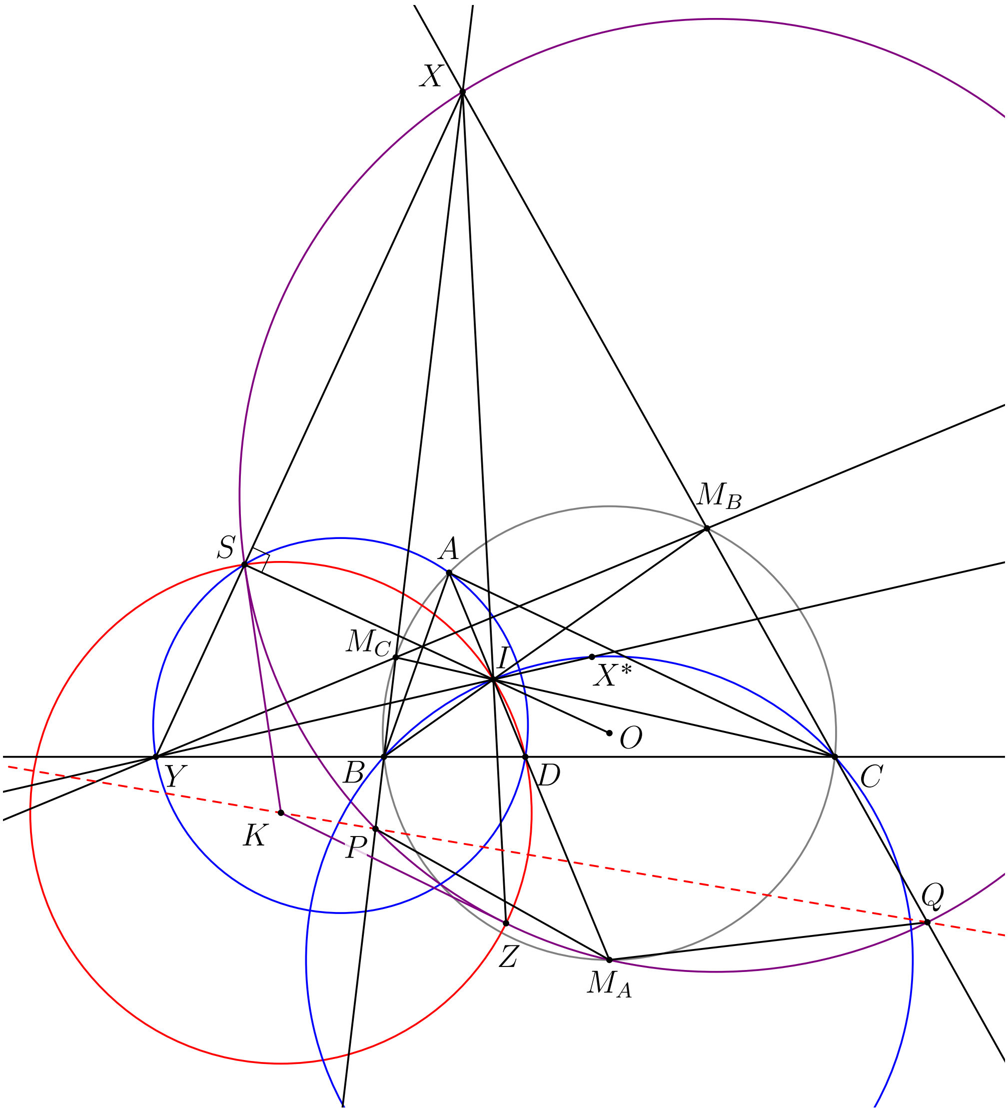


Proof. Let YI intersect AB and AC at E and F, respectively. From the parallel lines, we get that $\triangle BEP_1$ and $\triangle CQ_1F$ are homothetic with centre Y. Thus we have

$$\frac{YE}{YP_1} = \frac{YQ_1}{YF} \implies YP_1 \cdot YQ_1 = YE \cdot YF.$$

Moreover, AI bisects $\angle EAF$ and $YA = YI$ so the circle centred at Y with radius YA is the Apollonius circle of $\triangle AEF$ with respect to the feet of the internal and external angle bisectors at A. This gives $YE \cdot YF = YA^2$. Combining these results proves the claim. $\square$

Solution 3. As in Solution 1, let O be the circumcentre of $\triangle ABC$. Let XI intersect circle XSM_A again at $Z \neq X$ and let $Y = BC \cap M_BM_C$. Let X^* be the inverse of X in circle ABC. We will use the properties of Miquel point S noted at the top of Solution 1 and in addition, that S lies on line XY.



Claim 1. $YSAD$ is cyclic.

Proof. From $OM_A \perp BC$ and $YS \perp OS$ we have $\angle DYS = 180^\circ - \angle SOM_A$. From inverting collinear points A, I and M_A in circle ABC we get ASM_AO is cyclic which gives

$$\angle SOM_A = \angle SAM_A = \angle SAD \implies \angle SAD + \angle DYS = 180^\circ$$

proving the claim. □

Claim 2. X^* lies on circle BIC which has centre M_A.

Proof. This follows immediately from inverting circle $SBCX$ in circle ABC. □

Claim 3. Z lies on circle SID.

Proof. We have that

$$\angle IZS = \angle XM_AS = \angle OM_AS - \angle OM_AX = \angle M_AIO - \angle M_AX^*O = \angle DIO - \angle M_AX^*O$$

where in the penultimate step we inverted in circle ABC to get the angle equalities.

From Brocard's Theorem applied to cyclic quadrilateral BM_CM_BC, we get Y, I, and X^* collinear and $\angle YX^*O = 90°$. This gives that

$$\angle M_AX^*O = 90° - \angle IX^*M_A = 90° - \angle M_AIX^* = 90° - \angle AIY,$$

where the second equality is by Claim 2. We have that A and I are reflections in line M_BM_C. Hence,

$$90° - \angle AIY = 90° - \angle YAD = 90° - \angle YSD = \angle DSO$$

where the second step is by Claim 1, and in the last step we are using $OS \perp YS$. Putting these together,

$$\angle IZS = \angle DIO - \angle DSO = \angle IDS,$$

proving the claim. □

Let the tangents from S and Z to circle XSM_A intersect at K. Observe from the standard Ceva-Menelaus configuration,

$$-1 = (XY, XI; XB, XC) \stackrel{X}{=} (S, Z; P, Q).$$

This shows that K lies on line PQ. We then have

$$\angle ZKS = 180° - 2\angle SXZ = 2(90° - \angle SXI) = 2(180° - \angle SIZ),$$

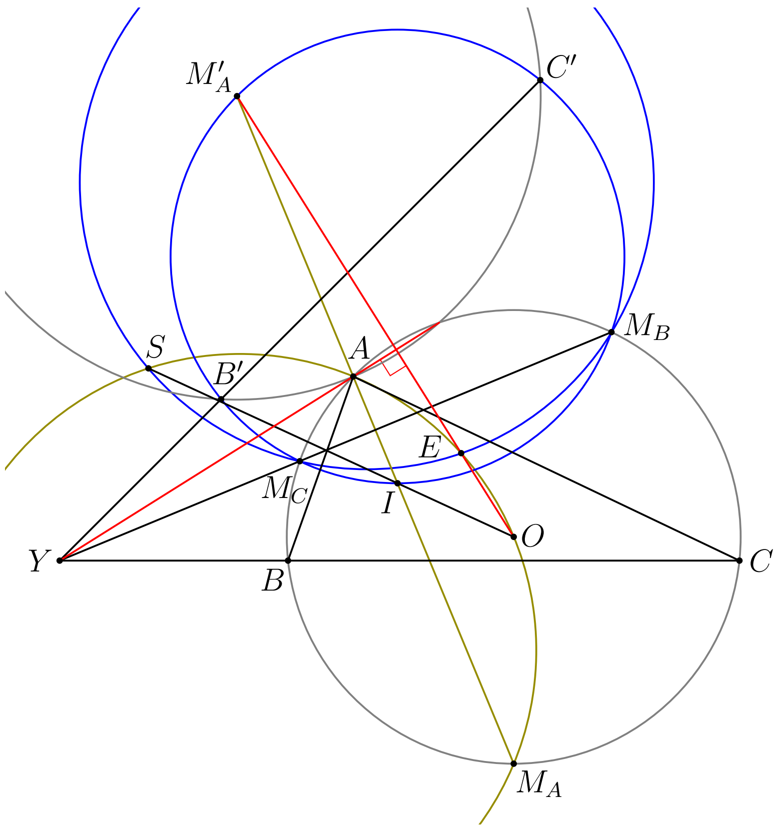
where we are using $\angle ISX = 90°$. As K lies on the perpendicular bisector of SZ, this is enough to show that K is the centre of circle $SIDZ$ completing the proof.

Solution 4. Solution 1 solves the problem by establishing $\frac{SP}{SQ} = \frac{IP}{IQ} = \frac{DP}{DQ}$, which implies that circle SID is an Apollonius circle with respect to P and Q. We demonstrate an alternate approach that only requires us to show *two* of the ratios $\frac{SP}{SQ}$, $\frac{IP}{IQ}$, and $\frac{DP}{DQ}$ to be equal. This can arise from missing some of the observations in Solution 1, for example not proving Claim 3.

Claim. Given we have shown two of the ratios listed above to be equal, it suffices to show that circle SID is orthogonal to circle SXM_A, which the same circle as SPQ.

Proof. Supposing we have shown the orthogonality, if $\frac{SP}{SQ} = \frac{IP}{IQ}$ or $\frac{SP}{SQ} = \frac{DP}{DQ}$, then we immediately have that circle SID is an Apollonius circle with respect to P and Q. If $\frac{IP}{IQ} = \frac{DP}{DQ}$ and S does not lie on the Apollonius circle $\mathcal{C}$ defined by this common ratio, then I and D lie on two distinct circles orthogonal to circle SPQ, namely circle SID and $\mathcal{C}$. This implies that I and D are inverses with respect to circle SPQ, which is a contradiction as both I and D lie inside circle SPQ. □

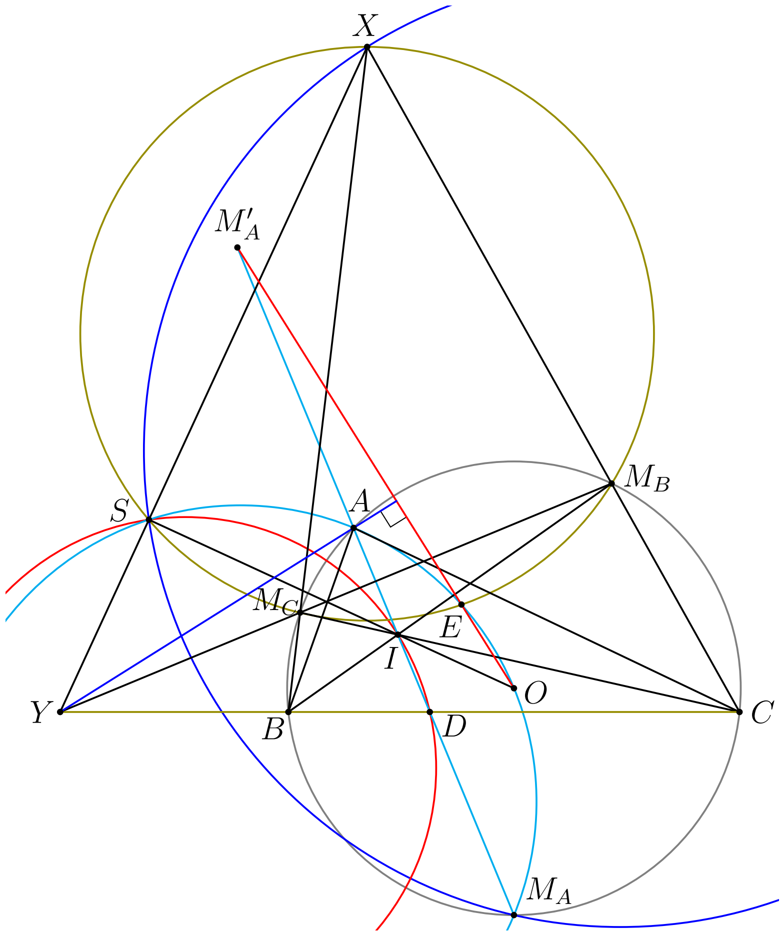
Throughout this solution, we will use the properties of S from the beginning of Solution 1. Define O and Y as in previous solutions, and let E be the second intersection of circles SOM_A and SM_BM_C.



Lemma. We have that $OE \perp AY$.

Proof. Let M'_A, B', and C' be the respective reflections of M_A, B, and C over line M_BM_C. As noted in Solution 3, A and I are reflections across M_BM_C. Because M_A is the centre of circle BIC, it follows that M'_A is the centre of circle $AB'C'$. On the other hand, Y lies on M_BM_C, so we have that $YB \cdot YC = YB' \cdot YC'$. Thus, Y lies on the radical axis of circles ABC and $AB'C'$, so $OM'_A \perp AY$. Finally, note that the inverses of circles SOM_A and SM_BM_C in circle ABC are line IM_A and circle IM_BM_C respectively, so E and M'_A are inverses in circle ABC. Thus, E lies on OM'_A and the lemma follows. □

Let $\mathcal{T}$ denote the composition of an inversion at S with radius $\sqrt{SI \cdot SO}$ with a reflection across line SI. By standard properties of the Miquel point, $\mathcal{T}$ swaps X and Y and any points Z_1 and Z_2 on circle ABC with $I \in Z_1Z_2$. Hence, $\mathcal{T}$ swaps the pairs (A, M_A), (B, M_B), (C, M_C), (O, I), and (X, Y). As $D = AI \cap BC$ and E is the intersection of circles SOM_A and SM_BM_C, we have that $\mathcal{T}(D) = E$. Thus, $\mathcal{T}$ maps circles SID and SXM_A to lines OE and AY, so by the Lemma, circles SID and SXM_A are orthogonal, as required.



G8. Let ABC be a triangle with $AB < AC < BC$, and let D be a point in the interior of segment BC. Let E be a point on the circumcircle of triangle ABC such that A and E lie on opposite sides of line BC and $\angle BAD = \angle EAC$. Let I, I_B, I_C, J_B, and J_C be the incentres of triangles ABC, ABD, ADC, ABE, and AEC, respectively.

Prove that I_B, I_C, J_B, and J_C are concyclic if and only if AI, I_BJ_C, and J_BI_C concur.

(Canada)

Solution 1. Let X be the intersection of I_BJ_C and J_BI_C. We will prove that, provided that $AB < AC < BC$, the following two conditions are equivalent:

(1) AX bisects $\angle BAC$;

(2) I_B, I_C, J_B, and J_C are concyclic.

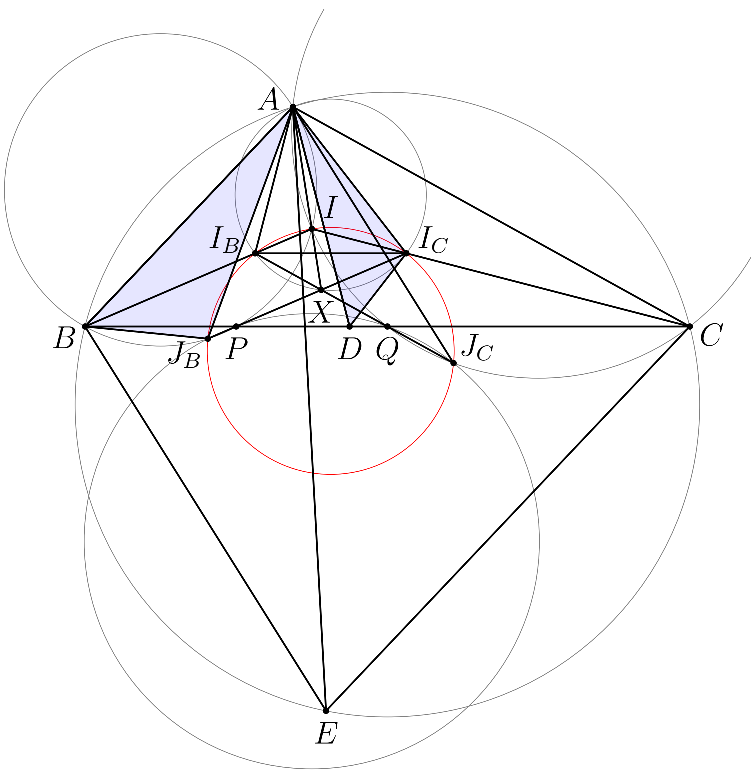
Let circles AIB and AIC meet BC again at P and Q, respectively. Note that $AB = BQ$ and $AC = CP$ because the centres of circles AIB and AIC lie on CI and BI, respectively. Thus, B, P, Q, and C are collinear in this order as $BQ + PC = AB + AC > BC$ by the triangle inequality.

Claim 1. Points P, J_B, and I_C are collinear, and points Q, I_B, and J_C are collinear.

Proof. We have that

$$\angle AJ_BB = 90^\circ + \frac{1}{2}\angle AEB = 90^\circ + \frac{1}{2}\angle ACB = \angle AIB = \angle APB,$$

so ABJ_BP is cyclic. As A is the centre of spiral similarity between ABE and ADC, it is also the centre of spiral similarity between ABJ_B and ADI_C. Hence, A is the Miquel point of self-intersecting quadrilateral BDI_CJ_B, so P lies on J_BI_C. Analogously, we have that Q lies on I_BJ_C. □



Throughout the rest of the solution, we will use directed angles.

Proof of (1) $\implies$ (2). We assume that (1) holds.

Claim 1 and the similarities $ABDI_B \sim AECJ_C$ and $ABEJ_B \sim ADCI_C$ tell us that

$$\measuredangle I_BXI_C = \measuredangle J_CQC + \measuredangle BPJ_B = \measuredangle J_CAC + \measuredangle BAJ_B = \measuredangle I_BAD + \measuredangle DAI_C = \measuredangle I_BAI_C,$$

so AI_BXI_C is cyclic. Also, as $X \in AI$, we have that

$$\measuredangle I_BAX = \measuredangle BAI - \measuredangle BAI_B = \measuredangle I_BAI_C - \measuredangle IB_AD = \measuredangle DAI_C.$$

Using these, we have that

$$\measuredangle I_BI_CP = \measuredangle I_BAX = \measuredangle DAI_C = \measuredangle BAJ_B = \measuredangle BPJ_B,$$

so $I_BI_C \parallel BC$. Hence,

$$\measuredangle I_BI_CJ_B = \measuredangle BPJ_B = \measuredangle BIJ_B = \measuredangle I_BIJ_B,$$

so $II_BJ_BI_C$ is cyclic. Analogously, we have that $II_CJ_CI_B$ is cyclic, so $I_BJ_BJ_CI_C$ is cyclic, thus proving (2). □

Proof of (2) $\implies$ (1). We assume that (2) holds.

Claim 2. Circles IBC, IJ_BI_C, and II_BJ_C are tangent at I.

Proof. Using the cyclic quadrilateral BIJ_BP, we have that

$$\measuredangle IBC = \measuredangle IBP = \measuredangle IJ_BP = \measuredangle IJ_BI_C.$$

As C, I_C, and I are collinear, the tangents to circles IJ_BI_C and IBC at I coincide, so circles IJ_BI_C and IBC are tangent at I. Analogously, circles II_BJ_C and IBC are tangent at I, so all three circles are tangent at I. □

Claim 3. Point I lies on circle $I_BJ_BJ_CI_C$.

Proof. Suppose that I does not lie on circle $I_BJ_BJ_CI_C$. Then the circles II_BJ_C, IJ_BI_C, and $I_BJ_BJ_CI_C$ are distinct. We apply the radical axis theorem to these three circles. By Claim 2, the radical axis of circles II_BJ_C and IJ_BI_C is the tangent to circle IBC at I. As I_BJ_C and J_BI_C intersect at X, IX must be tangent to circle IBC.

However, by Claim 1 we have that X is the intersection of PI_C and QI_B. As D lies on the interior of segment BC, I_B lies on the interior of segment BI and I_C lies on the interior of segment CI. Hence, I_B, P, Q, and I_C all lie on the perimeter of triangle IBC in this order, so X must be in the interior of triangle IBC. This means that IX cannot be tangent to circle BIC, so I must lie on circle $I_BJ_BJ_CI_C$. □

By Claims 2 and 3, circles II_BI_C and IBC are tangent, so $I_BI_C \parallel BC$. Since $I_BJ_BJ_CI_C$ is cyclic, we have that

$$\measuredangle PJ_BJ_C = \measuredangle I_CJ_BJ_C = \measuredangle I_CI_BJ_C = \measuredangle PQI_B = \measuredangle PQJ_C,$$

so PJ_BJ_CQ is cyclic. By the radical axis theorem on circles $AIPJ_B$, $AIQJ_C$, and PJ_BJ_CQ, we have that AI, I_BJ_C, and J_BI_C concur at X, thus proving (1). □

Solution 2. Let X be the intersection of I_BJ_C and J_BI_C. As in Solution 1, we will prove that conditions (1) and (2) are equivalent. To do so, we introduce the new condition:

(3) $I_BI_C \parallel BC$

and show that (3) is equivalent to both (1) and (2), provided that $AB < AC < BC$.

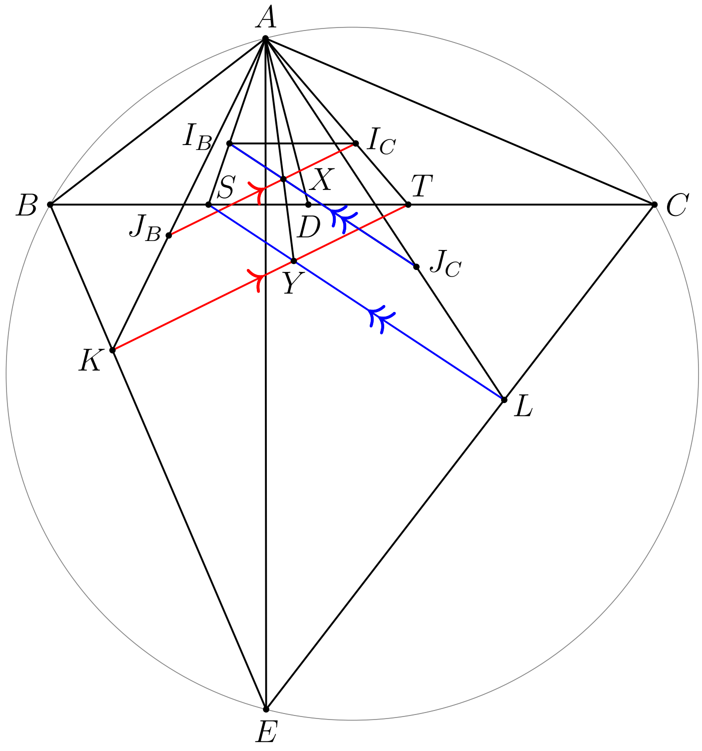
Note that $ABD \overset{+}{\sim} AEC$ and $ABE \overset{+}{\sim} ADC$, where $\overset{+}{\sim}$ denotes positive similarity. We will make use of the following fact.

Fact. For points P, P_1, P_2, P_3, and P_4, the positive similarities

$$PP_1P_2 \overset{+}{\sim} PP_3P_4 \quad \text{and} \quad PP_1P_3 \overset{+}{\sim} PP_2P_4$$

are equivalent.

Proof of (1) $\iff$ (3). Let AI_B and AI_C meet BC at S and T, respectively. Let AJ_B meet BE at K, AJ_C meet CE at L, and KT and SL meet at Y.



Claim 1. Line AY bisects $\angle BAC$.

Proof. Let Y' be the intersection of KT and the bisector of $\angle BAC$. As

$$\angle BAK = \frac{1}{2}\angle BAE = \frac{1}{2}\angle DAC = \angle TAC,$$

AY' also bisects $\angle KAT$. Hence, Y' is the foot of the bisector of $\angle KAT$ in triangle AKT. Using the Fact, we have that

$$\begin{aligned} ABE \overset{+}{\sim} ADC &\implies ABEK \overset{+}{\sim} ADCT \\ &\implies ABD \overset{+}{\sim} AKT \overset{+}{\sim} AEC \\ &\implies ABDS \overset{+}{\sim} AKTY' \overset{+}{\sim} AECL \\ &\implies ABEK \overset{+}{\sim} ASLY' \overset{+}{\sim} ADCT. \end{aligned}$$

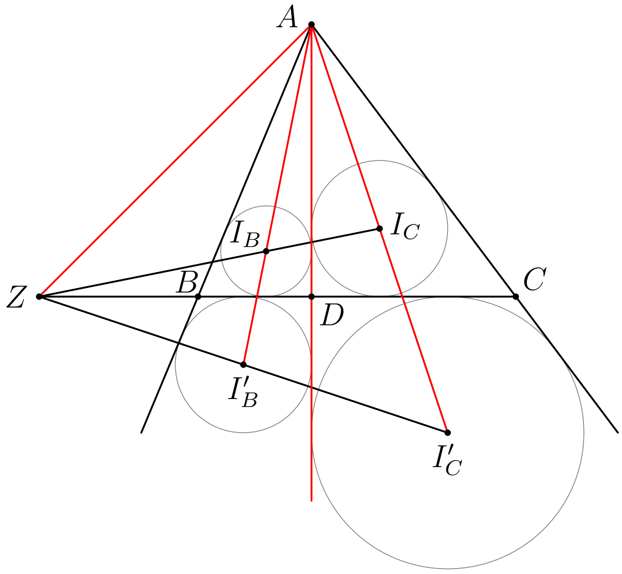
As K lies on BE, we have that Y' lies on SL, so $Y = Y'$ and AY bisects $\angle BAC$. □

We show that X lies on AY if and only if $I_BI_C \parallel BC$, which implies the equivalence of (1) and (3) by Claim 1. Let AY meet I_BJ_C and J_BI_C at X_1 and X_2, respectively. As ABD and AEC are similar, we have that $\frac{AI_B}{AS} = \frac{AJ_C}{AL}$, so $I_BJ_C \parallel SL$. Analogously, we have that $J_BI_C \parallel KT$. Hence, X_1 and X_2 coincide with X if and only if

$$\frac{AI_B}{AS} = \frac{AX_1}{AY} = \frac{AX_2}{AY} = \frac{AI_C}{AT},$$

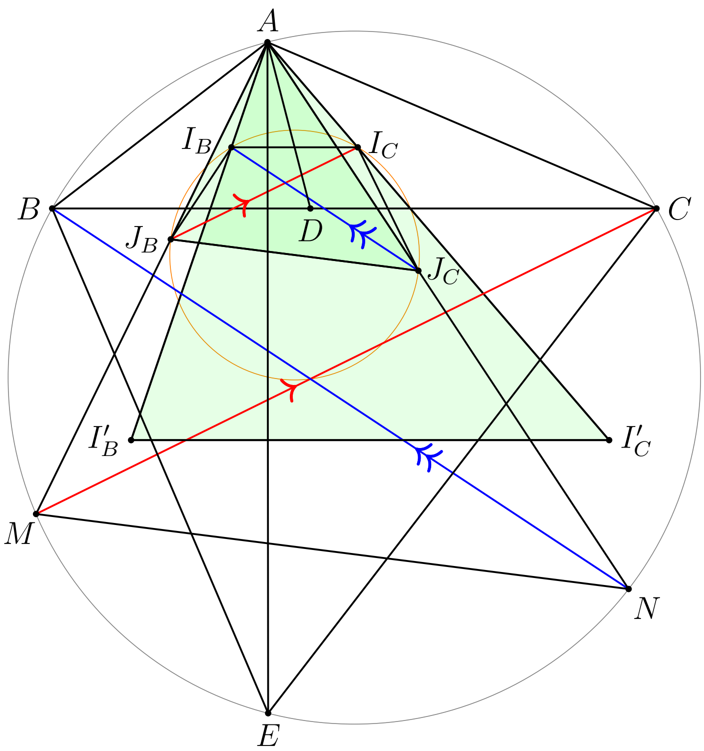
which is equivalent to $I_BI_C \parallel BC$. □

Proof of (2) $\iff$ (3). Let AJ_B and AJ_C meet circle ABC at M and N, respectively, and let I'_B and I'_C be the A-excentres of ABD and ADC, respectively.



Claim 2. Lines I_BI_C, $I'_BI'_C$, and BC are concurrent or pairwise parallel.

Proof. We work in the projective plane. Let I_BI_C and $I'_BI'_C$ meet BC at Z and Z', respectively. Note that Z is the intersection of the external common tangents of the incircles of ABD and ADC and AD is a common internal tangent of the incircles of ABD and ADC, so $(AD, AZ; AI_B, AI_C) = -1$. Applying the same argument to the A-excircles of ABD and ADC gives $(AD, AZ'; AI'_B, AI'_C) = -1$, which means that $Z = Z'$. Thus, I_BI_C, $I'_BI'_C$, and BC concur, possibly at infinity. □



Claim 3. Lines J_BI_C and CM are parallel, and lines I_BJ_C and BN are parallel.
Proof. Using the Fact, we have that

$$ABE \overset{+}{\sim} ADC \implies ABEJ_B \overset{+}{\sim} ADCI_C \implies AJ_BI_C \overset{+}{\sim} ABD.$$

Thus, $\angle(BD, J_BI_C) = \angle BAJ_B = \angle BCM$, so $J_BI_C \parallel CM$. Similarly, we have that $I_BJ_C \parallel BN$. □

Claim 4. The centre of spiral similarity between J_BJ_C and $I'_BI'_C$ is A.
Proof. As I_B and I'_B are respectively the incentre and A-excentre of triangle ABD, we have that $ABI'_B \overset{+}{\sim} AI_BD$. Using the similarity $ABD \overset{+}{\sim} AEC$, this means that $ABI'_B \overset{+}{\sim} AJ_CC$, so $AB \cdot AC = AI'_B \cdot AJ_C$ and $\angle BAI'_B = \angle J_CAC$. Similarly, we have that $AB \cdot AC = AJ_B \cdot AI'_C$ and $\angle BAJ_B = \angle I'_CAC$. Together, these imply that $AI'_B \cdot AJ_C = AJ_B \cdot AI'_C$ and $\angle J_BAJ_C = \angle I'_BAI'_C$, so $AJ_BJ_C \overset{+}{\sim} AI'_BI'_C$. □

We proceed using directed angles. By Claim 3, we have that $I_BJ_BJ_CI_C$ is cyclic if and only if

$$\begin{aligned}\measuredangle I_BI_CJ_B = \measuredangle I_BJ_CJ_B &\iff \measuredangle I_BI_CJ_B + \measuredangle MCB = \measuredangle I_BJ_CJ_B + \measuredangle MNB \\ &\iff \measuredangle(I_BI_C, BC) = \measuredangle(MN, J_BJ_C).\end{aligned}$$

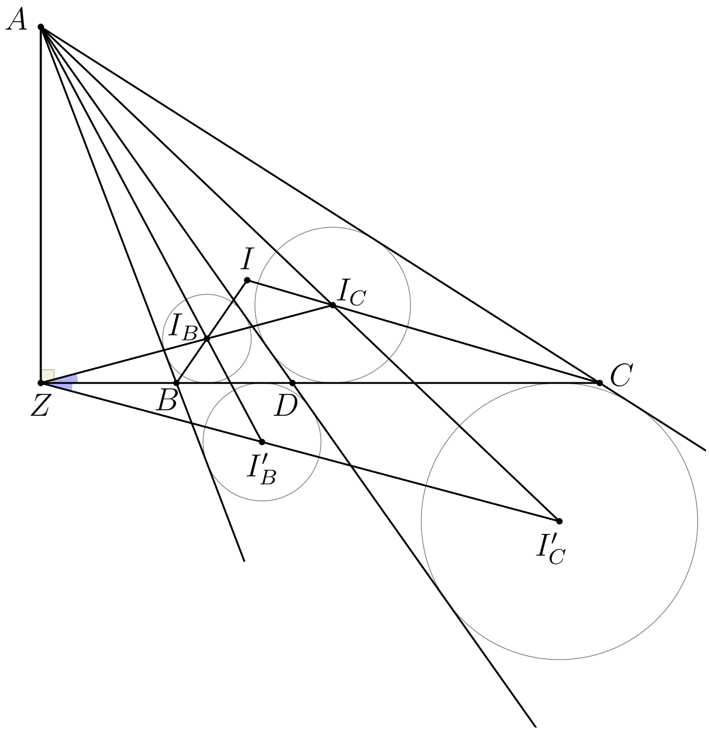
By Claim 4, we have that

$$\begin{aligned}\measuredangle(J_BJ_C, I'_BI'_C) &= \measuredangle J_BAI'_B \\ &= \measuredangle BAI_B + \measuredangle MAB \\ &= \measuredangle EAJ_C + \measuredangle MAB \\ &= \measuredangle NAC + \measuredangle MAB \\ &= \measuredangle(MN, BC),\end{aligned}$$

which is equivalent to $\measuredangle(BC, I'_BI'_C) = \measuredangle(MN, J_BJ_C)$. Thus, $I_BJ_BJ_CI_C$ is cyclic if and only if

$$\measuredangle(I_BI_C, BC) = \measuredangle(BC, I'_BI'_C). \qquad (*)$$

Suppose that I_BI_C is parallel to BC. By Claim 2, $I'_BI'_C$ is also parallel to BC, so we have that $\measuredangle(I_BI_C, BC) = \measuredangle(BC, I'_BI'_C) = 0°$. Thus, $(*)$ is satisfied, so $I_BJ_BJ_CI_C$ is cyclic.



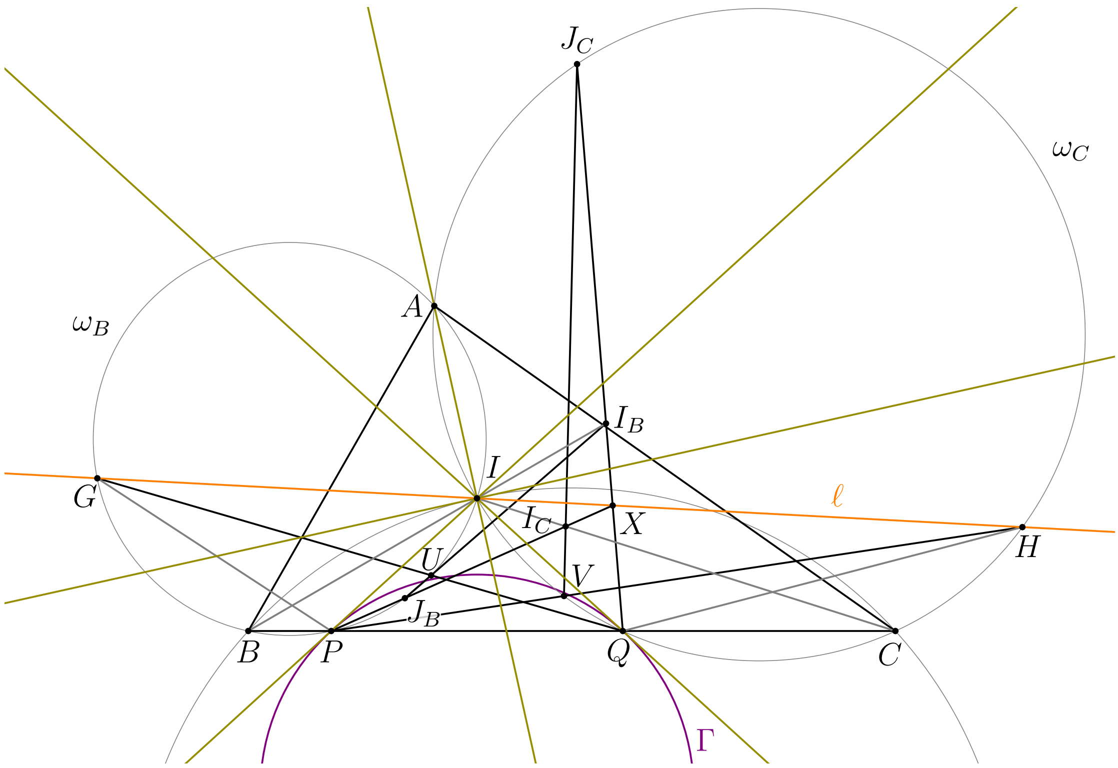
Suppose now that I_BI_C is not parallel to BC while $I_BJ_BJ_CI_C$ is cyclic. By Claim 2, I_BI_C, $I'_BI'_C$, and BC concur at a point Z. As I_B and I_C lie on segments BI and CI, Z must lie outside segment BC. Since A is the intersection of the common external tangents of the incircle and A-excircle of ABD and ZD is a common internal tangent of the incircle and A-excircle of ABD, we have that $(ZA, ZD; ZI_B, ZI'_B) = -1$. By $(*)$, ZD bisects $\angle I_BZI'_B$, so $\angle AZD = 90^\circ$: that is, Z is the foot from A to BC. But this implies that $\angle ABC$ or $\angle BCA$ is obtuse, contradicting the fact that $AB < AC < BC$. □

Comment. While we have written the solution using harmonic bundles for the sake of brevity, there are ways to prove Claim 2 and obtain the final contradiction without the use of projective geometry. Claim 2 can be proven using an application of Menelaus's theorem, and the final contradiction can be obtained using the fact that an excircle of a triangle is always larger than its incircle.

Solution 3. Let ω_B and ω_C denote circles AIB and AIC, respectively. Introduce P, Q and X as in Solution 1 and recall from Claim 1 in Solution 1 that P, J_B and I_C are collinear with J_B lying on ω_B. From this, we can define J_B and I_C in terms of X by $I_C = XP \cap CI$ and $J_B \neq P$ as the second intersection of line XP with ω_B. Similarly, we can define $I_B = XQ \cap BI$ and $J_C \neq Q$ as the second intersection of line XQ with ω_C. Note that this now detaches the definitions of points I_B, I_C, J_B, and J_C from points D and E.

Let ℓ be a line passing through I. We now allow X to vary along ℓ while fixing $\triangle ABC$ and points I, P, and Q. We use the definitions from above to construct I_B, I_C, J_B, and J_C. We will classify all cases where these four points are concyclic. Throughout the rest of the solution we use directed angles and directed lengths.

For nondegeneracy reasons, we exclude cases where $X = I$ and X lies on line BC, which means that I_B, $J_B \neq B$ and I_C, $J_C \neq C$. We also exclude the cases where ℓ is tangent to either ω_B or ω_C. Similar results hold in these cases and they can be treated as limit cases.



Claim 1. Line I_BJ_B passes through a fixed point on ω_B, and line I_CJ_C passes through a fixed point on ω_C as X varies on ℓ.

Proof. Let $U \neq J_B$ be the second intersection of I_BJ_B with ω_B. We have by the law of sines that

$$\frac{\sin \measuredangle IJ_BU}{\sin \measuredangle UJ_BB} = \frac{\sin \measuredangle IJ_BI_B}{\sin \measuredangle I_BJ_BB} = \frac{\sin \measuredangle J_BII_B}{\sin \measuredangle J_BBI_B} \cdot \frac{II_B}{I_BB} = \frac{\sin \measuredangle J_BIB}{\sin \measuredangle J_BBI} \cdot \frac{II_B}{I_BB} = \frac{\sin \measuredangle XPQ}{\sin \measuredangle XPI} \cdot \frac{II_B}{I_BB}.$$

We also have

$$\frac{II_B}{I_BB} = \frac{\sin \measuredangle IQI_B}{\sin \measuredangle I_BQB} \cdot \frac{|IQ|}{|BQ|} = \frac{\sin \measuredangle IQX}{\sin \measuredangle XQP} \cdot \frac{|IQ|}{|BQ|}.$$

Combining these and applying Ceva's Theorem in $\triangle PIQ$ with point X, we get

$$\frac{\sin \measuredangle IJ_BU}{\sin \measuredangle UJ_BB} = \frac{\sin \measuredangle XPQ}{\sin \measuredangle XPI} \cdot \frac{\sin \measuredangle IQX}{\sin \measuredangle XQP} \cdot \frac{|IQ|}{|BQ|} = \frac{\sin \measuredangle XIQ}{\sin \measuredangle XIP} \cdot \frac{|IQ|}{|BQ|} = \frac{\sin \measuredangle(\ell, IQ)}{\sin \measuredangle(\ell, IP)} \cdot \frac{|IQ|}{|BQ|},$$

which is independent of the choice of X on ℓ. As $\measuredangle IJ_BU + \measuredangle UJ_BB = \measuredangle IJ_BB = \measuredangle IAB$ is fixed, this is enough to show point U is fixed on ω_B.

Similarly, if we define $V \neq J_C$ to be the second intersection of I_CJ_C with ω_C, we get that V is fixed on ω_C. □

Let $G \neq X$ and $H \neq X$ be the second intersections of ℓ with ω_B and ω_C, respectively which exist as we are assuming ℓ is not tangent to either ω_B or ω_C.

Claim 2. Points U, G and Q are collinear and points V, H and P are collinear.

Proof. Taking $X = G$, we have $J_B = G$ and $I_B = XQ \cap BI$. Both of these points lie on line QG which, by Claim 1, shows that U, G, Q are collinear. Similarly, V, H, P are collinear. □

Claim 3. Points I_B, I_C, J_B, J_C are concyclic if and only if points P, Q, G, H are concyclic. In particular, this depends only on ℓ, not on the choice of X on ℓ.

Proof. We have that

$$\measuredangle I_CJ_BI_B = \measuredangle PJ_BU = \measuredangle PGU = \measuredangle PGQ$$
$$\measuredangle I_CJ_CI_B = \measuredangle VJ_CQ = \measuredangle VHQ = \measuredangle PHQ.$$

Thus $\measuredangle I_CJ_BI_B = \measuredangle I_CJ_CI_B \iff \measuredangle PGQ = \measuredangle PHQ$ which proves the claim. □

Claim 4. P, Q, G, H are concyclic if and only if $\ell \in \{IA, IP, IQ, t\}$ where t is the tangent to circle BIC at I.

Proof. When $\ell = IA$, we have $G = H = A$ so the cyclic condition from Claim 3 holds. Similarly, when $\ell = IP$ or $\ell = IQ$, $G = P$ or $H = Q$, respectively, so again the cyclic condition holds.

Now, consider the case where $\ell \notin \{IA, IP, IQ\}$. In this case it is straightforward to see that the four points P, Q, G, and H are distinct. We then have that $\measuredangle QPG = \measuredangle BPG = \measuredangle BIG$, so

$$PQGH \text{ concyclic} \iff \measuredangle QHG = \measuredangle QPG \iff \measuredangle QHG = \measuredangle BIG \iff QH \parallel BI.$$

We also have that $\measuredangle CQH = \measuredangle CIH$, so

$$\ell \text{ tangent to circle } BIC \iff \measuredangle CIH = \measuredangle CBI \iff \measuredangle CQH = \measuredangle CBI \iff QH \parallel BI.$$

Hence, in this case P, Q, G, H are concyclic if and only if ℓ is tangent to circle BIC, as claimed. □

We now revert to using points D and E to define points I_B, I_C, J_B, J_C, and X, returning to the original set-up.

Claim 5. Let Γ be the circle passing through P and Q that is tangent to IP and IQ, which exists as $IP = IQ = IA$. Then X lies on Γ. Furthermore, X lies on the same side of BC as A and does not lie on line BC.

Proof. We have that

$$\begin{aligned}\measuredangle XPI &= \measuredangle J_B PI = \measuredangle J_B AI = \measuredangle BAI - \measuredangle BAJ_B = \measuredangle J_B AJ_C - \measuredangle J_B AE \\ &= \measuredangle EAJ_C = \measuredangle J_C AC = \measuredangle J_C QC = \measuredangle XQP,\end{aligned}$$

so circle XPQ is tangent to IP. Similarly, circle XPQ is tangent to IQ, so X lies on Γ.

As D lies in the interior of segment BC, I_C lies in the interior of segment CI. Since X is the second intersection of PI_C with Γ and IP is tangent to Γ, X lies in the interior of $\overparen{PQ}$ on Γ on the same side of BC as A. This implies the second part of the claim. □

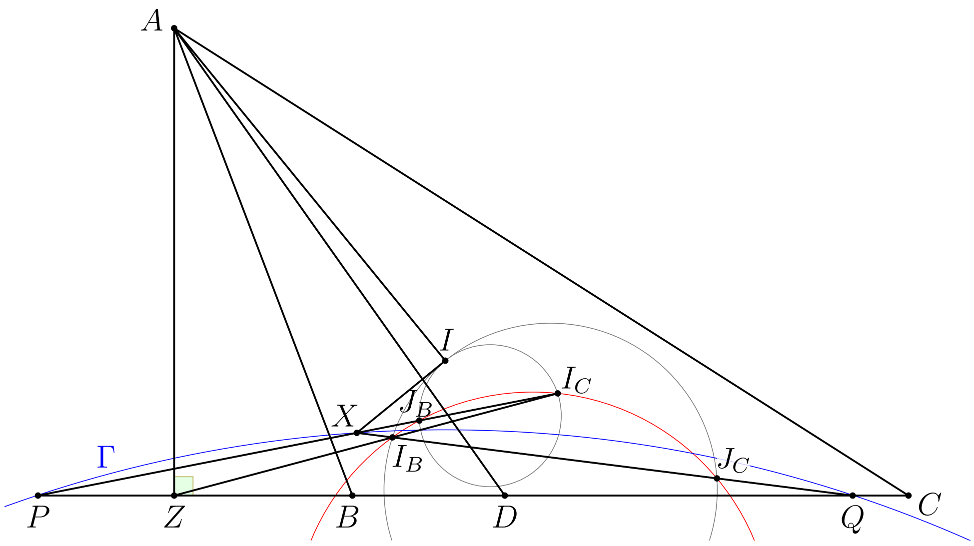
By Claim 5, we cannot have $\ell \in \{IP, IQ\}$ in the original problem. Furthermore, as shown in Claim 2 of Solution 1, we have that X lies inside triangle IBC, which means that $\ell \neq t$. Thus, the only remaining possibility in Claim 4 is $\ell = AI$. We then have

$$I_B I_C J_B J_C \text{ concyclic} \overset{\text{Claim 3}}{\Longleftrightarrow} PQGH \text{ concyclic} \overset{\text{Claim 4}}{\Longleftrightarrow} X \text{ lies on } AI,$$

finishing the problem.

Comment. The condition $AB < AC < BC$ is used in an essential way in the solutions. In Solution 1, it is used in the proof of Claim 3 to ensure that X lies in the interior of triangle IBC. In Solution 2, it is used in the final step to ensure that $\angle ABC$ and $\angle BCA$ cannot be obtuse. In Solution 3, it is used to exclude the case $\ell = t$. If the condition is removed, then the problem is no longer true: whenever $\angle ABC$ or $\angle BCA$ is obtuse, there exists a choice of D on BC such that $I_B J_B J_C I_C$ is cyclic but AI, $I_B J_C$, and $J_B I_C$ do not concur. This counterexample configuration can be constructed using Solution 3 by letting X be the intersection of t with Γ that lies on the same side of BC as A and constructing I_B, I_C, J_B, and J_C as described in the solution, from which we can reconstruct D.

Conversely, the problem holds whenever $\angle ABC$ and $\angle BCA$ are both not obtuse, as can be seen from Solution 2. This is thus the weakest possible condition on triangle ABC that is necessary for the problem to be true.



When X lies on the tangent to circle IBC at I, there is no contradiction in the proof of Claim 3 in Solution 1: circles $II_B J_C$ and $IJ_B I_C$ are distinct, and X is the radical centre of circles $II_B J_C$, $IJ_B I_C$, and $I_B J_B I_C J_C$. There is also no contradiction in the final step of Solution 2, and indeed $I_B I_C$ and BC intersect at the foot of the altitude from A to BC.

There are no configuration issues with the direction (1) $\Longrightarrow$ (2). This implication holds without any constraint on triangle ABC, and the proofs in Solutions 1 and 2 apply without any modification.

Number Theory

N1. Find all positive integers n with the following property: for all positive divisors d of n, we have that $d+1 \mid n$ or $d+1$ is prime.

(Ghana)

Answer: $n \in \{1, 2, 4, 12\}$.

Solution 1. It is easy to verify that $n = 1$, 2, 4, 12 all work. We must show they are the only possibilities. We write $n = 2^k m$, where k is a nonnegative integer and m is odd. Since $m \mid n$, either $m+1$ is prime or $m+1 \mid n$.

In the former case, since $m+1$ is even it must be 2, so $n = 2^k$. If $k \geqslant 3$, we get a contradiction, since $8 \mid n$ but $9 \nmid n$. Hence $k \leqslant 2$, so $n \in \{1, 2, 4\}$.

In the latter case, we have $m+1 \mid 2^k m$ and $m+1$ coprime to m, and hence $m+1 \mid 2^k$. This means that $m+1 = 2^j$ with $2 \leqslant j \leqslant k$ (since $j = 1$ gives $m = 1$, which was considered earlier).

Then we have $2^k + 1 \nmid n$: since $2^k + 1$ is odd, it would have to divide m but is larger than m. Hence, by the condition of the problem, $2^k + 1$ is prime. If $k = 2$, j must be 2 as well, and this gives the solution $n = 12$. Also, $2^{k-1} + 1 \nmid n$ for $k > 2$: since it is odd, it would have to divide m. However, we have no solutions to $2^{k-1} + 1 \mid 2^j - 1$ with $j \leqslant k$: the left-hand side is greater than the right unless $j = k$, when the left-hand side is just over half the right-hand side.

Since we have $2^k \mid n$ and $2^k + 1 \nmid n$, and $2^{k-1} \mid n$ and $2^{k-1} + 1 \nmid n$, we must have $2^k + 1$ and $2^{k-1} + 1$ both prime. However, $2^a + 1$ is a multiple of three if a is odd, so we must have $2^k + 1 = 3$ (impossible as this gives $k = 1$) or $2^{k-1} + 1 = 3$, which gives $j = k = 2$, whence $n = 12$.

Solution 2. We proceed as in Solution 1 as far as determining that $n = 2^k(2^j - 1)$ with $j \leqslant k$.

Now, we have $2^j \mid n$ but $2^j + 1 \nmid n$, as it is odd and does not divide $2^j - 1$. Thus $2^j + 1$ is prime. The theory of Fermat primes tells us we must have $j = 2^h$ with $h > 0$.

Then $2^{2^h} - 1$ is congruent to 3 or 6 (modulo 9) depending on whether h is odd or even, respectively. In particular it is not divisible by 9, so $n = 2^k(2^{2^h} - 1)$ is not divisible by 9; so we must have $k \leqslant 2$, since if $k \geqslant 3$ then $8 \mid n$ but $9 \nmid n$ with 9 not prime.

Solution 3. Let p be the smallest integer not dividing n. Since $p-1$ is a divisor of n, p must be a prime. Let $1 \leqslant r \leqslant p-1$ be the remainder of n modulo p. Since $p - r < p$, we have $p - r \mid n$, so we may consider the divisor $d = \frac{n}{p-r}$.

Since $p \mid n - r$, we have $p \mid n + p - r$, whence $p \mid d+1$. Thus $d+1 \nmid n$; so it must be prime. On the other hand, this prime is divisible by p, so we conclude $d+1 = p$, which means that $n = (p-1)(p-r)$.

Then from $p-2, p-3 \mid n$ we get $(p-2)(p-3) \mid 2(p-r)$, from which we find

$$(p-2)(p-3) \leqslant 2(p-r) \leqslant 2(p-1).$$

Solving this quadratic inequality gives $p \leqslant 5$, which means that $n \in \{1, 2, 4, 8, 12, 16\}$. Of this set, $n = 8$ and $n = 16$ are not solutions.

Solution 4. We suppose that n is not 1 or 2.

Since $n \mid n$ and $n+1 \nmid n$, we know that $n+1$ is prime. Thus it is odd, so $2 \mid n$; as $n > 2$, we have $\frac{n}{2} \mid n$ and $\frac{n}{2} + 1 \nmid n$, so $\frac{n}{2} + 1$ is prime. Thus it is also odd, so $4 \mid n$.

We must then have $\frac{n}{4} + 1 \mid n$ or $\frac{n}{4} + 1$ prime.

In the former case, $\frac{n}{4} \mid 4(\frac{n}{4}+1)-n$, so $\frac{n}{4}+1 \mid 4$. This means that $n=4$ or $n=12$.

In the latter case, $\frac{n}{4}+1$ must be odd if $n \neq 4$. Thus we have $n=8m$ where $2m+1$, $4m+1$, $8m+1$ are all prime; $n=8$ does not work, so $3 \mid m$ (otherwise one of those numbers would be divisible by 3). Thus $24 \mid n$, so $25 \mid n$ as 25 is not prime.

Now suppose that p is the least positive integer not dividing n: as in Solution 3 we know that p is prime, and what we have done so far shows that $p \geqslant 7$. If $p^2-1=(p-1)(p+1)$ is the product of coprime integers less than p, it divides n, and p^2 is not prime so also divides n (a contradiction); $p-1$ and $p+1$ are even and have no common factor higher than 2, so all odd prime power divisors of their product are less than p and the only case where p^2-1 is not a product of coprime integers less than p is when one of $p-1$ and $p+1$ is a power of 2, say 2^m (with $m \geqslant 3$). If $p=2^m-1$, then $3p-1=4(3\times 2^{m-2}-1)$ and $3\times 2^{m-2}-1$ is an odd integer less than p, so $3p-1 \mid n$ and so $3p \mid n$. Finally, if $p=2^m+1$, then m is even and $2p-1=2^{m+1}+1$ is a multiple of 3; the only case where it is a power of 3 is when $m=2$, but we have $m \geqslant 3$, so $2p-1$ is a product of coprime integers less than p and again we have a contradiction.

Solution 5. As in Solution 4, we deduce that if $n>2$ then n must be even. We write $n=2\cdot 3^k\cdot r$, where k is a nonnegative integer and $3 \nmid r$.

Since r and $2r$ are both different and nonzero modulo 3, one of them must be congruent to 2 modulo 3. We'll say that it is ar, where $a \in \{1,2\}$.

Since $ar \mid n$, we must have that $ar+1$ is either prime or a factor of n. In the first case, $ar+1=3$ because $3 \mid ar+1$, and so $n=2\cdot 3^k\cdot r$, where $r=2/a$ is 1 or 2. Noting that we must have $k \leqslant 1$ (else $9 \mid n$ but $10 \nmid n$), we can examine cases to deduce that $n \in \{2,4,12\}$ are the only possibilities.

Otherwise, $ar+1 \mid n$. Since $ar+1$ is coprime to r, we must in fact have that $ar+1 \mid 2\cdot 3^k$, and since $3 \mid ar+1$ by assumption we deduce that $k \geqslant 1$. In particular, 3^k+1 is an even number that is at least 4, so is not prime and must divide n. As it is coprime to 3, we must in fact have $3^k+1 \mid 2r$.

Let q_1 and q_2 be such that $q_1(ar+1)=2\cdot 3^k$ and $q_2(3^k+1)=2r$. We have that $q_1ar<2\cdot 3^k$ and $q_2 3^k<2r$, and multiplying these together gives $q_1q_2a<4$.

If $a=2$ then $q_1=q_2=1$, so $2r+1=2\cdot 3^k$, which is not possible (considering both sides modulo 2).

If $a=1$ then r must be equivalent to 2 modulo 3, so $q_2(3^k+1)=2r$ gives that q_2 is equivalent to 1 modulo 3, whence $q_2=1$. So we deduce that $2r=3^k+1$. Thus, we deduce that $q_1(3^k+3)=4\cdot 3^k$, which rearranges to give $3^{k-1}(4-q_1)=q_1$, whence $3^{k-1} \leqslant q_1 < 4$ and so $k \leqslant 2$. We can examine cases to deduce that $n=12$ is the only possibility.

N2. Determine all finite, nonempty sets $\mathcal{S}$ of positive integers such that for every $a, b \in \mathcal{S}$ there exists $c \in \mathcal{S}$ with $a \mid b + 2c$.

(Netherlands)

Answer: The possible sets are $\mathcal{S} = \{t\}$ and $\mathcal{S} = \{t, 3t\}$ for any positive integer t.

Solution 1. Without loss of generality, we may divide all elements of $\mathcal{S}$ by any common factor, after which they cannot all be even. As $a \nmid b + 2c$ for a even and b odd, the elements of $\mathcal{S}$ are all odd.

We now divide into three cases:

Case 1: $|\mathcal{S}| = 1$.

The set $\mathcal{S} = \{t\}$ clearly works.

Case 2: $|\mathcal{S}| = 2$.

Say $\mathcal{S} = \{r, s\}$ with $r < s$, so either $s \mid r + 2r$ or $s \mid r + 2s$, and in either case $s \mid 3r$. We cannot have $s = 3r/2$ as we assumed that r is odd, so $s = 3r$ and $\mathcal{S} = \{r, 3r\}$, which clearly works by examining cases for a and b.

Case 3: $|\mathcal{S}| \geqslant 3$.

If all elements of $\mathcal{S}$ are odd then for any $b, c \in \mathcal{S}$, $b + 2c \not\equiv b \pmod 4$. If $a \mid b + 2c$ with $a \equiv b \pmod 4$, this means there exists k with $b + 2c = ka$ and $k \equiv 3 \pmod 4$, so $k \geqslant 3$. If a is the greatest element of $\mathcal{S}$ and $b < a$, we have $b + 2c < 3a$, a contradiction. Thus when a is the greatest element, no $b < a$ has $b \equiv a \pmod 4$ (and thus all elements other than the greatest are congruent modulo 4).

Let d and e be the largest and second largest element of $\mathcal{S}$ respectively. Let $f \neq d, e$ be any other element of $\mathcal{S}$. There is some $c \in \mathcal{S}$ with $e \mid f + 2c$, and $e \not\equiv f + 2c \pmod 4$, so $f + 2c \geqslant 3e$, so $c > e$. Since e is the second largest element of $\mathcal{S}$, $c = d$, so $e \mid f + 2d$, and this holds for all $f \in \mathcal{S}$ with $f < e$, but can only hold for at most one such f. So $|\mathcal{S}| \leqslant 3$.

Hence the elements of $\mathcal{S}$ are $d > e > f$, and by the discussion above without loss of generality we may suppose these elements are all odd, $e \equiv f \pmod 4$ and $d \not\equiv e \pmod 4$. We have above that $e \mid f + 2d$. Furthermore, there exists some $c \in \mathcal{S}$ with $d \mid f + 2c$, and $c \neq d$ as $d > f$ so $d \nmid f$, so $c \leqslant e$; as $f + 2e < 3e$, we have $e > d/3$. Since $f + 2c$ is odd and $f + 2c < 3d$, we have $f + 2c = d$.

Subcase 3.1: $c = f$.

Here $d = 3f$ and $e \mid f + 2d = 7f$. As $e > f$ and $e \equiv f \pmod 4$, we have $e = 7f/3$ and the elements are some multiples of $\{3, 7, 9\}$. But $a = 7$ and $b = 9$ have no corresponding value of c.

Subcase 3.2: $c = e$.

Here $d = f + 2e$ and $e \mid f + 2d = 3f + 4e$ so $e \mid 3f$. But this is not possible with $e > f$ and $e \equiv f \pmod 4$.

Solution 2. As in Solution 1, we reduce to the case where all elements of $\mathcal{S}$ are odd. Since all one-element sets satisfy the given conditions, we show that if $|\mathcal{S}| \geqslant 2$, then $|\mathcal{S}| = 2$ and $\mathcal{S} = \{t, 3t\}$ for some positive integer t.

Let d be the largest element. For any $e \in \mathcal{S}$ with $e \neq d$ there must be a $f \in \mathcal{S}$ such that $d \mid e + 2f$. This implies $2f \equiv -e \pmod d$, hence $2f \equiv d - e \pmod d$. Now $d - e$ is even (because all elements in $\mathcal{S}$ are odd) and d is odd, so $\frac{d-e}{2}$ is an integer and we have $f \equiv \frac{d-e}{2} \pmod d$. Further, $0 < \frac{d-e}{2} < d$, while we must also have $0 < f \leqslant d$, so $f = \frac{d-e}{2}$. We conclude that for any $e \in \mathcal{S}$ with $e \neq d$ the integer $\frac{d-e}{2}$ is also in $\mathcal{S}$ and not equal to d.

Denote by $e_1 < e_2 < \cdots < e_k < d$ the elements of $\mathcal{S}$, where $k \geqslant 1$. Then $\frac{d-e_1}{2} > \frac{d-e_2}{2} > \cdots > \frac{d-e_k}{2}$ are also elements of $\mathcal{S}$, none of them equal to d. Hence we must have $e_1 = \frac{d-e_k}{2}$ and

$e_k = \frac{d-e_1}{2}$, so $2e_1+e_k = d = 2e_k+e_1$. We conclude $e_1 = e_k$, so $k = 1$, and also $d = 2e_k+e_1 = 3e_1$. Hence $\mathcal{S} = \{e_1, 3e_1\}$ for some positive integer e_1.

Solution 3. As in Solution 1, we reduce to the case where all elements of $\mathcal{S}$ are odd. Since all one-element sets satisfy the given conditions, we show that if $|\mathcal{S}| \geqslant 2$, then $|\mathcal{S}| = 2$ and $\mathcal{S} = \{t, 3t\}$ for some positive integer t.

Let d be the largest element, and let $e \in \mathcal{S}$ be any other element. We will say that $x \in \mathcal{S} \pmod d$ if the unique element y in $\{1, \ldots, d\}$ such that $x \equiv y \pmod d$ is an element of $\mathcal{S}$. Note that by the choice of d being the largest element, if $x \neq d$, then $x \not\equiv 0 \pmod d$. The given condition implies that if $b \in \mathcal{S}$, then $-\frac{b}{2} \in \mathcal{S} \pmod d$. Repeating this gives $-\frac{b}{2} \in \mathcal{S} \Rightarrow \frac{b}{4} \in \mathcal{S} \pmod d$, and by iterating, we have $b \in \mathcal{S} \Rightarrow \frac{b}{(-2)^k} \in \mathcal{S} \pmod d$ for all k. Since d is odd, there is some g such that $(-2)^g \equiv 1 \pmod d$, so by setting $k = g-1$, we get that

$$\text{for all } d \neq e \in \mathcal{S}, -2e \in \mathcal{S} \pmod d.$$

Now, if $e > \frac{d}{2}$, then $-2e \in \mathcal{S} \pmod d$ and $d - 2e < 0$, so $2d - 2e \in \mathcal{S}$, contradicting the lack of even elements. Then $e < \frac{d}{2}$ for any $e \in \mathcal{S} \setminus \{d\}$, so we have $e \in \mathcal{S} \Rightarrow d - 2e \in \mathcal{S}$. Since $d - 2e \neq d$, we must have $d - 2e < \frac{d}{2}$, which rearranges to $e > \frac{d}{4}$.

Let $\lambda \in (0, 1)$ be a positive real number and suppose we have proved that $e > \lambda d$ for any $e \in \mathcal{S} \setminus \{d\}$. Then $d - 2e > \lambda d$, which rearranges to $e < \frac{(1-\lambda)d}{2}$. Then $d - 2e < \frac{(1-\lambda)d}{2}$, which rearranges to $e > \frac{(1+\lambda)d}{4}$. Defining $\lambda_0 = \frac{1}{4}$ and $\lambda_i = \frac{1+\lambda_{i-1}}{4}$ for $i \geqslant 1$, we have shown that for all $e \in \mathcal{S} \setminus \{d\}$ and all λ_i, $e > \lambda_i d$. Now note that the sequence λ_i is increasing and bounded above by $\frac{1}{3}$, so it converges to some limit ℓ, which satisfies $\ell = \frac{1+\ell}{4}$, so $\ell = \frac{1}{3}$. Hence $e \geqslant \frac{d}{3}$, but then $d - 2e \geqslant \frac{d}{3}$ implies $e \leqslant \frac{d}{3}$, so e must be $\frac{d}{3}$, and we are done.

Comment. We can finish Solution 3 alternatively as follows: after showing that if $e \in \mathcal{S} \setminus \{d\}$ then $d - 2e \in \mathcal{S} \setminus \{d\}$, note that

$$(d-2e) - \frac{d}{3} = \frac{2d}{3} - 2e = -2\left(e - \frac{d}{3}\right).$$

So consider $e \in \mathcal{S} \setminus \{d\}$ maximising $|e - \frac{d}{3}|$. If $e \neq \frac{d}{3}$, them the above shows that $|(d-2e) - \frac{d}{3}| > |e - \frac{d}{3}|$, which is a contradiction. Thus $\mathcal{S} \setminus \{d\}$ is empty or equal to $\{\frac{d}{3}\}$, which completes the proof.

N3. Determine all sequences $a_1,\ a_2,\ \ldots$ of positive integers such that, for any pair of positive integers $m \leqslant n$, the arithmetic and geometric means

$$\frac{a_m + a_{m+1} + \cdots + a_n}{n-m+1} \quad \text{and} \quad (a_m a_{m+1} \cdots a_n)^{\frac{1}{n-m+1}}$$

are both integers.

(Singapore)

Answer: The only such sequences are the constant sequences (which clearly work).

Solution 1. We say that an integer sequence $b_1,\ b_2,\ \ldots$ is *good* if for any pair of positive integers $m \leqslant n$, the arithmetic mean $\frac{b_m+b_{m+1}+\cdots+b_n}{n-m+1}$ is an integer. Then the condition in the question is equivalent to saying that the sequences (a_i) and $(\nu_p(a_i))$ for all primes p are good.

Claim 1. If (b_i) is a good sequence, then $n-m \mid b_n - b_m$ for all pairs of integers $m,\ n$.

Proof. This follows from $n-m$ dividing $b_m + b_{m+1} + \cdots + b_{n-1}$ and $b_{m+1} + b_{m+2} + \cdots + b_n$, and then taking the difference. □

Claim 2. If (b_i) is a good sequence where some integer b occurs infinitely many times, then (b_i) is constant.

Proof. Say $b_{n_1},\ b_{n_2},\ b_{n_3},\ \ldots$ are equal to b. Then for all m, we have that $b - b_m = b_{n_j} - b_m$ is divisible by infinitely many different integers $n_j - m$, so it must be zero. Therefore the sequence is constant. □

Now, for a given prime p, we look at the sequence $(\nu_p(a_i))$. Let $k = \nu_p(a_1)$. Then Claim 1 tells us that $a_1 \equiv a_{np^{k+1}+1} \pmod{p^{k+1}}$ for all n, which implies that $\nu_p(a_{np^{k+1}+1}) = k$ for all n. We now have that k appears infinitely many times in this good sequence, so by Claim 2, the sequence $(\nu_p(a_i))$ is constant. This holds for all primes p, so (a_i) must in fact be constant.

Solution 2. As in Claim 1 of Solution 1, we have that $a_{i+r} \equiv a_i \pmod{r}$, which tells us that the sequence a_i is periodic modulo p with period p. Also, by a similar argument, we have that a_{i+r}/a_i is the r^{th} power of a rational number.

Now suppose that for some $i \not\equiv j \pmod{p}$ we have $a_i,\ a_j \not\equiv 0 \pmod{p}$. As p and $p-1$ are coprime, we can find some $i' \equiv i \pmod{p}$, $j' \equiv j \pmod{p}$ such that $p-1 \mid i'-j'$. Then $a_{i'}/a_{j'}$ is a perfect $(p-1)^{\text{th}}$ power, so

$$a_{i'} = tu^{p-1}, \quad a_{j'} = tv^{p-1}$$

for some positive integers $t,\ u,\ v$ not divisible by p. By Fermat's little theorem, u^{p-1} and v^{p-1} must be 1 modulo p. So we must have

$$a_i \equiv a_{i'} \equiv t \equiv a_{j'} \equiv a_j \pmod{p}.$$

Thus all values of a_i that are not divisible by p are congruent modulo p.

For the sum of p consecutive values to be divisible by p, this means that all the a_i are congruent modulo p. Since this is true for all primes p, the sequence must therefore be constant.

Solution 3. Fix an arbitrary index m. First, we show that a_m divides a_n for sufficiently large n. Let n be sufficiently large that $n > \nu_p(a_m) + m$ for every prime p. By Claim 1 of Solution 1, we have

$$\nu_p(a_m) \equiv \nu_p(a_n) \pmod{n-m}.$$

Since $\nu_p(a_m) < n - m$, it follows that $\nu_p(a_m) \leqslant \nu_p(a_n)$. This holds for every prime p, so $a_m \mid a_n$.

Next, suppose that there is some index k such that a_m does not divide a_k. By the previous, there is a maximal such k. Then $a_{k+1}, a_{k+2}, \ldots$ are all divisible by a_m. But now applying the first condition gives

$$a_m \mid a_k + a_{k+1} + \cdots + a_{k+a_m-1},$$

so a_m divides a_k, a contradiction. Therefore every term a_n is divisible by a_m.

As m was arbitrary, we now have $a_m \mid a_n$ and vice versa for all m, n. So the sequence must be constant.

N4. Determine all positive integers a and b such that there exists a positive integer g such that $\gcd(a^n + b, b^n + a) = g$ for all sufficiently large n.

(Indonesia)

Answer: The only solution is $(a, b) = (1, 1)$.

Solution 1. It is clear that we may take $g = 2$ for $(a, b) = (1, 1)$. Supposing that (a, b) satisfies the conditions in the problem, let N be a positive integer such that $\gcd(a^n + b, b^n + a) = g$ for all $n \geqslant N$.

Lemma. We have that $g = \gcd(a, b)$ or $g = 2\gcd(a, b)$.

Proof. Note that both $a^N + b$ and $a^{N+1} + b$ are divisible by g. Hence

$$a(a^N + b) - (a^{N+1} + b) = ab - b = a(b - 1)$$

is divisible by g. Analogously, $b(a - 1)$ is divisible by g. Their difference $a - b$ is then divisible by g, so g also divides $a(b - 1) + a(a - b) = a^2 - a$. All powers of a are then congruent modulo g, so $a + b \equiv a^N + b \equiv 0 \pmod{g}$. Then $2a = (a + b) + (a - b)$ and $2b = (a + b) - (a - b)$ are both divisible by g, so $g \mid 2\gcd(a, b)$. On the other hand, it is clear that $\gcd(a, b) \mid g$, thus proving the Lemma. □

Let $d = \gcd(a, b)$, and write $a = dx$ and $b = dy$ for coprime positive integers x and y. We have that

$$\gcd\left((dx)^n + dy, (dy)^n + dx\right) = d\gcd\left(d^{n-1}x^n + y, d^{n-1}y^n + x\right),$$

so the Lemma tells us that

$$\gcd\left(d^{n-1}x^n + y, d^{n-1}y^n + x\right) \leqslant 2$$

for all $n \geqslant N$. Defining $K = d^2xy + 1$, note that K is coprime to each of d, x, and y. By Euler's theorem, for $n \equiv -1 \pmod{\varphi(K)}$ we have that

$$d^{n-1}x^n + y \equiv d^{-2}x^{-1} + y \equiv d^{-2}x^{-1}(1 + d^2xy) \equiv 0 \pmod{K},$$

so $K \mid d^{n-1}x^n + y$. Analogously, we have that $K \mid d^{n-1}y^n + x$. Taking such an n which also satisfies $n \geqslant N$ gives us that

$$K \mid \gcd(d^{n-1}x^n + y, d^{n-1}y^n + x) \leqslant 2.$$

This is only possible when $d = x = y = 1$, which yields the only solution $(a, b) = (1, 1)$.

Solution 2. After proving the Lemma, one can finish the solution as follows.

For any prime factor p of $ab + 1$, p is coprime to a and b. Take an $n \geqslant N$ such that $n \equiv -1 \pmod{p - 1}$. By Fermat's little theorem, we have that

$$a^n + b \equiv a^{-1} + b = a^{-1}(1 + ab) \equiv 0 \pmod{p},$$
$$b^n + a \equiv b^{-1} + a = b^{-1}(1 + ab) \equiv 0 \pmod{p},$$

then p divides g. By the Lemma, we have that $p \mid 2\gcd(a, b)$, and thus $p = 2$. Therefore, $ab + 1$ is a power of 2, and a and b are both odd numbers.

If $(a, b) \neq (1, 1)$, then $ab + 1$ is divisible by 4, hence $\{a, b\} = \{-1, 1\} \pmod{4}$. For odd $n \geqslant N$, we have that

$$a^n + b \equiv b^n + a \equiv (-1) + 1 = 0 \pmod{4},$$

then $4 \mid g$. But by the Lemma, we have that $\nu_2(g) \leqslant \nu_2(2\gcd(a, b)) = 1$, which is a contradiction. So the only solution to the problem is $(a, b) = (1, 1)$.

N5. Let $\mathcal{S}$ be a finite nonempty set of prime numbers. Let $1 = b_1 < b_2 < \cdots$ be the sequence of all positive integers whose prime divisors all belong to $\mathcal{S}$. Prove that, for all but finitely many positive integers n, there exist positive integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} = \left\lfloor \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \right\rfloor.$$

(Croatia)

Solution 1. If $\mathcal{S}$ has only one element p, then $b_i = p^{i-1}$ and we can easily find $a_1, \ldots, a_n$ with $2 = \left\lfloor \sum_{i=0}^{n-1} \frac{1}{p^i} \right\rfloor = \sum_{i=0}^{n-1} \frac{a_i}{p^{i-1}}$ by taking $a_1 = a_2 = \cdots = a_{n-1} = 1$ and choosing $a_n = p^{n-1} - (p + p^2 + \ldots + p^{n-2})$.

More generally, observe that the sum of $\frac{1}{b_i}$ over all i is

$$\begin{aligned}\sum_i \frac{1}{b_i} &= \prod_i \left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \ldots\right) \\ &= \prod_{p \in \mathcal{S}} \frac{p}{p-1}.\end{aligned}$$

In particular, if n is large enough, then

$$\left\lfloor \sum_{j=1}^{n} \frac{1}{b_j} \right\rfloor = \left\lfloor \prod_{p \in \mathcal{S}} \frac{p}{p-1} \right\rfloor.$$

For the remainder of the proof, we will only consider n large enough that this equality holds.

Next, we handle the special case $\mathcal{S} = \{2, 3\}$, for which this product is 3. Start by setting

$$a_i = \begin{cases} 1, & \text{if } 2b_i \leqslant b_n; \\ 2, & \text{if } 2b_i > b_n. \end{cases}$$

Then,

$$\sum_{\substack{i \leqslant n \\ \nu_3(b_i) = t}} \frac{a_i}{b_i} = \begin{cases} \frac{2}{3^t}, & \text{if } b_n \geqslant 3^t; \\ 0, & \text{otherwise.} \end{cases}$$

As a result,

$$\begin{aligned}\sum_{i \leqslant n} \frac{a_i}{b_i} &= \sum_{\substack{t \geqslant 0 \\ 3^t \leqslant b_n}} \frac{2}{3^t} \\ &= 3 - \frac{1}{3^T}\end{aligned}$$

where T is the largest $t \geqslant 0$ with $3^t \leqslant b_n$. Thus, increasing a_j by one (where $b_j = 3^T$) gives a sequence of a_i that works.

Otherwise, we may assume that $|\mathcal{S}| > 1$ and $\mathcal{S} \neq \{2, 3\}$, which means that the product $\prod_{p \in \mathcal{S}} \frac{p}{p-1}$ is not an integer. Indeed,

- if $|\mathcal{S}| > 2$ then 2 divides the denominator at least twice and so divides the denominator of the overall fraction;
- if $|\mathcal{S}| = 2$ and $2 \notin \mathcal{S}$ then 2 divides the denominator and not the numerator;

- if $\mathcal{S} = \{2, p\}$ then the product is $2p/(p-1)$ which is not an integer for $p > 3$.

It follows that for some fixed $\alpha > 0$, we have that

$$\left\lceil \prod_{p \in \mathcal{S}} \frac{p}{p-1} \right\rceil = \prod_{p \in \mathcal{S}} \frac{p}{p-1} + \alpha,$$

from which it follows that

$$\left\lceil \sum_{i=1}^{n} \frac{1}{b_i} \right\rceil - \sum_{i=1}^{n} \frac{1}{b_i} > \alpha.$$

It will now suffice to prove the following claim.

Claim. Suppose that n is large enough, and let e_p be the largest nonnegative integer such that $p^{e_p} \leqslant b_n$. Let $M = \prod_{p \in \mathcal{S}} p^{e_p}$. If u is a positive integer such that $u/M > \alpha$, then there exist nonnegative integers a_i such that

$$\sum_i \frac{a_i}{b_i} = \frac{u}{M}.$$

The problem statement follows after replacing a_i with $a_i + 1$ for each i.

To prove this, choose some constant c such that $\sum_{p \in \mathcal{S}} p^{-c} < \alpha$, and suppose n is large enough that $p^c < b_n$ for each $p \in \mathcal{S}$; in particular, $p^c \mid M$ with M defined as above.

For each $p \in \mathcal{S}$, let i_p be such that $b_{i_p} = p^{e_p}$ and choose the smallest nonnegative integer a_{i_p} satisfying

$$p^{e_p - c} \;\Big|\; a_{i_p} \left(\frac{M}{p^{e_p}} \right) - u.$$

Such an a_{i_p} must exist and be at most $p^{e_p - c}$; indeed, $\frac{M}{p^{e_p}}$ is an integer coprime to p, so we can take a_{i_p} to be equal to u times its multiplicative inverse modulo $p^{e_p - c}$. The sum of the contributions to the sum from the a_{i_p} is at most

$$\sum_{p \in \mathcal{S}} \frac{p^{e_p - c}}{p^{e_p}} = \sum_{p \in \mathcal{S}} p^{-c} < \alpha.$$

So, we have

$$\frac{u}{M} = \sum_{p \in \mathcal{S}} \frac{a_{i_p}}{p^{e_p}} + \frac{r}{\prod_{p \in \mathcal{S}} p^c},$$

where r is an integer because of our choice of a_{i_p} and r is nonnegative because of the bound on u. Simply choose $a_i = r$ where $b_i = \prod_{p \in \mathcal{S}} p^c$ to complete the proof.

Solution 2. We reduce to the claim as in Solution 1, and provide an alternative approach for constructing the a_i.

Let $p_0 \in \mathcal{S}$ be the smallest prime in $\mathcal{S}$. Let $z_0 = u/M$. We construct a sequence z_0, z_1, z_2, ... and values of a_i by the following iterative process: to construct z_{j+1},

- select the largest prime $p \in \mathcal{S}$ dividing the denominator of z_j, and let μ be the number of times p divides the denominator of z_j;
- choose the largest ν such that $p_0^{\nu} p^{\mu} \leqslant b_n$, and let $i \leqslant n$ be such that $b_i = p_0^{\nu} p^{\mu}$;
- choose $0 \leqslant a_i < p$ such that the denominator of $z_k - a_i/b_i$ has at most $\mu - 1$ factors of p, and let $z_{k+1} = z_k - a_i/b_i$;
- continue until p_0 is the only prime dividing the denominator of z_k.

Note that we can always choose a_i in step 3; by construction, $z_k b_i$ has no factors of p in its denominator, so must be realised as an element of $\mathbb{Z}_p$.

Each time we do this, $b_i > M/p_0$ by construction, so

$$\frac{a_i}{b_i} < \frac{pp_0}{M} \leqslant \frac{p_0 p_1}{M},$$

where p_1 is the largest prime in $\mathcal{S}$. And the number of times we do this operation is at most

$$\sum_{\substack{p\in\mathcal{S}\\ p>p_0}} e_p \leqslant |\mathcal{S}| \log_2(M),$$

so the sum of the a_i/b_i we have assigned is at most $|\mathcal{S}|p_0p_1 \log_2(M)/M$.

Choose n large enough that $\log_2(M)/M < \alpha$; after subtracting the above choices of a_i/b_i from u/M, we have a quantity of the form $r/p_0^{e_{p_0}}$, where r is an integer by construction and r is positive by the above bounds. Simply set $a_i = r$ where $b_i = p_0^{e_{p_0}}$ to complete the proof.

Solution 3. As in Solution 1, we may handle $|\mathcal{S}| = 1$ and $\mathcal{S} = \{2, 3\}$ separately; otherwise, we can define α as we did in that solution. Also define e_p to be the largest nonnegative integer such that $p^{e_p} \leqslant b_n$ as we did in Solution 1.

We will show that, for n sufficiently large, we may choose some $j \leqslant n$, and positive integers a_i, such that

$$\sum_{i\neq j} \frac{a_i}{b_i} - \sum_{i\neq j} \frac{1}{b_i} < \alpha.$$

and all $\frac{a_i}{b_i}$ are integer multiples of $\frac{1}{b_j}$. We then set a_j to be the least positive integer such that the sum on the left is an integer, which will obviously have the required value.

Concretely, choose j such that $b_j = \prod_{p\in\mathcal{S}} p^{\lfloor e_p/|\mathcal{S}|\rfloor}$, which is less than b_n by construction. For $i \neq j$, set $a_i = b_i/\gcd(b_i, b_j)$. We have

$$\sum_{i\neq j} \frac{a_i}{b_i} - \sum_{i\neq j} \frac{1}{b_i} < \sum_{\substack{i\neq j\\ a_i>1}} \frac{a_i}{b_i}.$$

If $a_i > 1$, then there must be some $p \in \mathcal{S}$ for which $p^{\lfloor e_p/|\mathcal{S}|\rfloor+1} \mid b_i$, and so

$$\frac{a_i}{b_i} = \frac{1}{\gcd(b_i, b_j)} \leqslant \frac{1}{p^{\lfloor e_p/|\mathcal{S}|\rfloor}} < \frac{p}{b_n^{1/|\mathcal{S}|}},$$

where the last inequality follows from the fact that $p^{e_p+1} > b_n$.

Now $n \leqslant \prod_{p\in\mathcal{S}}(\log_p(b_n) + 1) \leqslant (2\log b_n)^{|\mathcal{S}|}$, so

$$\sum_{\substack{i\neq j\\ a_i>1}} \frac{a_i}{b_i} \leqslant \frac{(2\log b_n)^{|\mathcal{S}|}}{b_n^{1/|\mathcal{S}|}},$$

and so we can choose n large enough that this quantity is less than α, as required.

N6. Let n be a positive integer. We say that a polynomial P with integer coefficients is *n-good* if there exists a polynomial Q of degree 2 with integer coefficients such that $Q(k)(P(k)+Q(k))$ is never divisible by n for any integer k.

Determine all integers n such that every polynomial with integer coefficients is an n-good polynomial.

(France)

Answer: The set of such n is any $n > 2$.

Solution 1. First, observe that no polynomial is 1-good (because $Q(X)(P(X)+Q(X))$ always has roots modulo 1) and the polynomial $P(X) = 1$ is not 2-good (because $Q(X)(Q(X)+1)$ is always divisible by 2).

Now, if P is d-good with some Q, then $Q \cdot (P+Q)$ has no roots mod d. Therefore, it certainly has no roots mod n for $d \mid n$, so P must be n-good. Consequently, it suffices to show that all polynomials are n-good whenever n is an odd prime, or $n = 4$.

We start by handling the case $n = 4$. We will construct a Q such that $Q(X)$ is never divisible by 4 and $Q(X) + P(X)$ is always odd; this will clearly show that P is 4-good. Note that any function modulo 2 must be either constant or linear – in other words, there are $a, b \in \{0, 1\}$ such that $P(X) = aX + b \bmod 2$ for all X. If $a = 0$ then set $Q(X) = 4X^2 + b + 1$, and if $a = 1$ then set $Q(X) = X^2 + b + 1$; in all cases, Q will satisfy the required properties.

It remains to prove that any polynomial is p-good, where p is an odd prime. We will prove that for any *function* f defined mod p, there is a quadratic Q with no roots mod p such that $Q(x) \neq f(x)$ mod p for all x; the statement about P then follows with f replaced by $-P$. For the remainder of the proof, we will consider all equalities modulo p.

Suppose that a function f not satisfying the above exists; in other words, f has the property that for any quadratic Q with no roots mod p, there is some x such that $Q(x) = f(x)$. Without loss of generality, we may assume that f has no roots mod p. To see why, suppose that $f(u) = 0$ for some u, and let g be the function such that $g(x) = f(x)$ for $x \neq u$ and $g(u) = 1$. For any Q with no roots, we know that there is some $x \neq u$ such that $P(x) = f(x)$, and so $P(x) = g(x)$ for that choice of x. In particular, g is also not p-good.

Now, suppose first that there is some nonzero t such that t is not in the image of f. Then we may take $Q(X) = pX^2 + t$; this quadratic is never equal to f and is never zero. Thus, f must be surjective onto the nonzero residues mod p. There are p choices for X and $p-1$ nonzero residues mod p, so there must be some $x_1 \neq x_2$ mod p such that $f(x_1) = f(x_2)$, and f is a bijection from the set of residues mod p not equal to x_2 to the set of nonzero residues mod p.

Now, note that we may choose any b and c with b nonzero and replace $f(X)$ with $g(X) = f(bX+c)$; if there were some Q with no roots such that $Q(x) \neq g(x)$ for all x, then $Q(X/b - c/b)$ would work for f. Choose b and c such that $bx_1 + c = 1$ and $bx_2 + c = -1$; such b and c must exist (we may take $b = 2/(x_1 - x_2)$ and $c = (x_1 + x_2)/(x_2 - x_1)$). Renaming g to f, we see that we may assume $f(1) = f(-1)$.

Let r' be a quadratic nonresidue mod p. Choose $y \neq 0$ such that $f(y) = (1 - r')f(0)$, which must exist as the right hand side is nonzero and $1 - r'$ is not equal to 1. Choose $r = y^2/r'$, which is a quadratic nonresidue.

Consider $\phi(X) = f(X)/(X^2 - r)$. By definition, $\phi(1) = \phi(-1)$ and $\phi(0) = \phi(y)$, so there are no more than $p-2$ values in the image of ϕ. Choose some nonzero a not in the image of ϕ, so $f(X)/(X^2 - r)$ is never equal to a. The quadratic $Q(X) = a(X^2 - r)$ is never zero and also never equal to $f(X)$, which completes the proof.

Comment. In fact, there is no need to pass from polynomials P to functions f, as any function mod p is a polynomial. Concretely, instead of passing from f to g, we would have instead replaced $P(X)$ with $P(X)+1-(X-u)^{p-1}$, which is a polynomial that is unchanged except at $X=u$.

Solution 2. Given f a function mod p such that f is surjective onto the nonzero elements of $\mathbb{Z}/p\mathbb{Z}$ and $f(1)=f(-1)$, we provide an alternative approach to construct a nonzero quadratic $Q(X)$ such that $Q(X)\neq f(X)$. Let r be the smallest quadratic nonresidue mod p (so $r-1$ is a square) and let a vary over the nonzero elements mod p; we will show that it is possible to choose $Q_a(X)=a(X^2-r)$ for some choice of a. Note that any quadratic of this form will be nowhere zero.

Suppose that no such Q_a works. Then, for each a, there exists x such that $a(x^2-r)=f(x)$. We may assume that $x\neq -1$, as if the equality holds for $x=-1$ then it also holds for $x=1$. However, $a(x^2-r)=f(x)$ implies $a=f(x)/(x^2-r)$, so $f(x)/(x^2-r)$ must be a surjection from $\{x\neq -1\}$ to the set of nonzero a, and so this is a bijection. In particular, for each a, there exists a unique x_a such that $f(x_a)=a(x_a^2-r)$.

We now have

$$\begin{aligned}\prod_{t\neq 0} t &= \prod_{a\neq 0} f(x_a)\\ &= \prod_{a\neq 0} a \prod_{a\neq 0}(x_a^2-r)\\ &= \prod_{a\neq 0} a \prod_{x\neq -1}(x^2-r)\end{aligned}$$

where the first equality follows because f is surjective onto the nonzero residues mod p, and the second equality follows from the definition of x_a. The two products cancel, which means that $\prod_{x\neq -1}(x^2-r)=1$.

However, we also get

$$\prod_{x\neq -1}(x^2-r)=(-r)(1-r)\left(\prod_{x=2}^{(p-1)/2}(x^2-r)\right)^2.$$

However, this is a contradiction as $-r(1-r)=r(r-1)$, which is not a quadratic residue (by our choice of r).

Comment. By Wilson's theorem, we know that the product of the nonzero elements mod p is -1; however, this fact was not necessary for the solution so we chose to present the solution without needing to state it.

Comment. One can in fact show that

$$\prod_{x\neq -1}(x^2-r)=\frac{-4r}{1-r}.$$

To do this, note that the polynomial $X^{\frac{p-1}{2}}-1$ has the $\frac{p-1}{2}$ quadratic residues as roots, so we have

$$\prod_{s \text{ quad. res.}}(X-s)=X^{\frac{p-1}{2}}-1$$

and so

$$\prod_{x\neq 0}(X-x^2)=(X^{\frac{p-1}{2}}-1)^2.$$

Since r is a quadratic nonresidue, by Euler's criterion $r^{\frac{p-1}{2}}=-1$, and the result follows.

Therefore, one can replace the condition that r is the smallest quadratic nonresidue with the condition that r is a quadratic nonresidue not equal to $-\frac{1}{3}$ (which is possible for all $p\geqslant 3$).

Solution 3. As in Solution 1, we will reduce to the case of p being an odd prime and f being a function mod p with no roots which is surjective onto the set of nonzero residues mod p, although we make no assumption about the values of x_1 and x_2 with $f(x_1) = f(x_2)$.

We will again consider quadratics of the form $Q_{a,b,c}(X) = aR(bX+c)$, where $R(X) = X^2-r$ for an arbitrary fixed quadratic nonresidue r, a and b are nonzero mod p, and c is any residue mod p.

For each fixed b and c, there must be n pairs (a, x) such that $aR(bx + c) = f(x)$, because there must be exactly one value of a for each x. If any a appears in no such pair then we are done, so assume otherwise. In other words, there must be exactly one a such that there are two such x, and for all other a there is only one such x.

Thus, for each (b, c), there is exactly one unordered pair $\{x_1, x_2\}$ such that for some a we have $f(x_i) = aR(bx_i + c)$; in other words, there is exactly one unordered pair $\{x_1, x_2\}$ such that $f(x_1)/R(bx_1 + c) = f(x_2)/R(bx_2 + c)$.

Now, we show that for each unordered pair $\{x_1, x_2\}$ there must be at least one pair (b, c) such that $f(x_1)/R(bx_1 + c) = f(x_2)/R(bx_2 + c)$. Indeed, let $t = f(x_1)/f(x_2)$. There must be some x'_1, x'_2 such that $R(x'_1)/R(x'_2) = t$; this is because $R(X)$ and $tR(X)$ both take $\frac{p+1}{2}$ nonzero values mod p, so the intersection must be nonempty by the pigeonhole principle. Choosing b and c such that $bx_1 + c = x'_1$ and $bx_2 + c = x'_2$ gives the claim.

Note further that if (b, c) and $\{x_1, x_2\}$ satisfy the relation, then the same is true for $(-b, -c)$ and $\{x_1, x_2\}$ because $R(bx + c) = R(-bx - c)$. Since b is nonzero, this means that each pair $\{x_1, x_2\}$ corresponds to at least two pairs (b, c). However, since there are $p(p - 1)$ pairs (b, c) with b nonzero and $p(p-1)/2$ unordered pairs $\{x_1, x_2\}$, each $\{x_1, x_2\}$ must correspond to exactly two pairs (b, c) and $(-b, -c)$ for some (b, c).

Now, since the image of f has only $p - 1$ elements, there must be some x_1, x_2 such that $f(x_1) = f(x_2)$. Choose any b, c such that $bx_1 + c = -(bx_2 + c)$, so $R(bx_1 + c) = R(bx_2 + c)$ and so $f(x_1)/R(bx_1 + c) = f(x_2)/R(bx_2 + c)$. There is such a pair b, c for any nonzero b, so there are at least $p - 1$ such pairs, and this quantity is greater than 2 for $p \geqslant 5$.

Finally, for the special case that $p = 3$, we observe that there must be at least one allowed value for $Q(x)$ for each x, so there must exist such a quadratic Q by Lagrange interpolation.

Comment. We may also handle the case $p = 3$ as follows. Recall that we may assume f is nonzero and surjective onto $\{1, 2\}$ mod 3, so the image of f must be $(1, 1, 2)$ or $(1, 2, 2)$ in some order. Without loss of generality $f(1) = f(2)$, so we either have $(f(0), f(1), f(2)) = (1, 2, 2)$ or $(2, 1, 1)$. In the first case, take $Q(X) = 2X^2 + 2$, and in the second case take $Q(X) = X^2 + 1$.

In some sense, this is equivalent to the Lagrange interpolation approach, as in each case the polynomial $Q(X)$ can be determined by Lagrange interpolation.

Solution 4. Again, we reduce to the case of p being an odd prime and f being a function mod p; we will show that there is a quadratic which is nowhere zero such that $Q(x) = f(x)$ has no root. We can handle the case of $p = 3$ separately as in Solution 3, so assume that $p \geqslant 5$.

We will prove the following more general statement: let $p \geqslant 5$ be a prime and let $\mathcal{A}_1$, $\mathcal{A}_2$, $\ldots$, $\mathcal{A}_p$ be subsets of $\mathbb{Z}/p\mathbb{Z}$ with $|\mathcal{A}_i| = 2$ for all i. Then there exists a polynomial $Q \in \mathbb{Z}/p\mathbb{Z}[X]$ of degree at most 2 such that $Q(i) \notin \mathcal{A}_i$ for all i. Indeed, applying this statement to the sets $\mathcal{A}_i = \{0, f(i)\}$ (and adding pX^2 if necessary) produces a quadratic Q satisfying the desired property.

Choose the coefficients of Q uniformly at random from $\mathbb{Z}/p\mathbb{Z}$, and let T be the random variable denoting the number of i for which $Q(i) \in \mathcal{A}_i$. Observe that for $k \leqslant 3$, we have

$$\mathbb{E}\left[\binom{T}{k}\right] = 2^k \binom{p}{k} p^{-k}.$$

To see why, let $k \leqslant 3$. If $\mathcal{S} \subseteq \mathbb{Z}/p\mathbb{Z}$ has size k and $(a_i)_{i\in\mathcal{S}}$ is a k-tuple, the probability that $Q(i) = a_i$ on $\mathcal{S}$ is equal to p^{-k}; for $k = 3$ this follows by Lagrange interpolation, and for $k < 3$

it follows from the $k = 3$ case by summing. The expectation is therefore equal to the number of $\mathcal{S} \subseteq \mathbb{Z}/p\mathbb{Z}$ of size k times the probability that $Q(i) \in \mathcal{A}_i$ for each $i \in \mathcal{S}$, which is equal to the right hand side as each $\mathcal{A}_i$ has size 2.

Now, observe that we have the identity $(t-1)(t-3)(t-4) = -12 + 12\binom{t}{1} - 10\binom{t}{2} + 6\binom{t}{3}$, so

$$\begin{aligned}\mathbb{E}[(T-1)(T-3)(T-4)] &= -12 + 12\mathbb{E}\left[\binom{T}{1}\right] - 10\mathbb{E}\left[\binom{T}{2}\right] + 6\mathbb{E}\left[\binom{T}{3}\right] \\ &= -12 + 12 \cdot 2 - 10 \cdot 2\left(1 - \frac{1}{p}\right) + 6 \cdot \frac{4}{3}\left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right) \\ &= -\frac{4}{p} + \frac{16}{p^2}.\end{aligned}$$

This is negative for $p \geqslant 5$. Because $(t-1)(t-3)(t-4) \geqslant 0$ for all integers $t > 0$, it then follows that $T = 0$ with positive probability, which implies that there must exist some Q with $Q(i) \notin \mathcal{A}_i$ for all i, as desired.

Comment. We do not have much freedom to choose a different polynomial in place of $R(T) = (T-1)(T-3)(T-4)$ in this argument. Indeed, it can be shown (by comparing coefficients of $\binom{T}{K}$) that if R has degree at most 3, then the expected value of $R(T)$ tends to $\frac{1}{3}(R(4) + 2R(1))$ as p tends to infinity, so R must have both 1 and 4 as roots. In particular, R must be of the form $R(T) = (T-1)(T-4)(T-d)$ for some $d \geqslant 3$, and if $d < 4$ then the argument works for any p with $p > 4/(4-d)$.

N7. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Let $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be a function satisfying the following property: for m, $n \in \mathbb{Z}_{>0}$, the equation

$$f(mn)^2 = f(m^2)f(f(n))f(mf(n))$$

holds if and only if m and n are coprime.

For each positive integer n, determine all the possible values of $f(n)$.

(Japan)

Answer: All numbers with the same set of prime factors as n.

Common remarks. We refer to the given property as $P(m,n)$. We use the notation $\operatorname{rad}(n)$ for the radical of n: the product of the distinct primes dividing n.

Solution 1. We start with a series of straightforward deductions:

- From $P(1,1)$, we have $f(1)^2 = f(1)f(f(1))^2$, so $f(1) = f(f(1))^2$.
- From $P(1, f(1))$, we have $f(f(1))^2 = f(1)f(f(f(1)))f(f(f(1)))$, so $f(f(f(1))) = 1$.
- From $P(1, f(f(1)))$, we have $f(f(f(1)))^2 = f(1)f(f(f(f(1))))f(f(f(f(1))))$, which simplifies to $1 = f(1)^3$, so $f(1) = 1$.
- From $P(1,n)$ we deduce $f(n) = f(f(n))$ for all n.
- From $P(m,1)$ we deduce $f(m) = f(m^2)$ for all m.
- Simplifying $P(m,n)$, we have that

  $$f(mn)^2 = f(m)f(n)f(mf(n))$$

  if and only if m and n are coprime; refer to this as $Q(m,n)$.
- From $Q(m, f(n))$, we have that $f(mf(n)) = f(m)f(n)$ if and only if m and $f(n)$ are coprime; refer to this as $R(m,n)$.

Claim. If $f(a) = 1$, then $a = 1$.

Proof. If $a \neq 1$, then $Q(a,a)$ gives $f(a)^2 \neq f(a)^2 f(af(a))$. If $f(a) = 1$, then both sides simplify to 1, a contradiction. □

Claim. If $n \neq 1$ then $\gcd(n, f(n)) \neq 1$.

Proof. If $\gcd(n, f(n)) = 1$, then $Q(f(n), n)$ gives $f(nf(n))^2 = f(n)^3$, and $Q(n, f(n))$ gives $f(nf(n))^2 = f(n)^2 f(nf(n))$, which together yield $f(n) = 1$ for a contradiction. □

Claim. For all n we have $\operatorname{rad}(n) \mid f(n)$.

Proof. For any prime $p \mid n$, write $n = p^v n'$ with $p \nmid n'$. From $Q(p^v, n')$ we have $f(n)^2 = f(p^v)f(n')f(p^v f(n'))$. Since $\gcd(p^v, f(p^v)) \neq 1$, it follows that $p \mid f(p^v)$, so $p \mid f(n)$, and thus $\operatorname{rad}(n) \mid f(n)$. □

Claim. If n is coprime to $f(k)$, then $f(n)$ is coprime to $f(k)$.

Proof. From $Q(f(k), n)$ we have $f(nf(k))^2 = f(k)f(n)f(f(k)f(n))$; applying $R(n,k)$ to the LHS, we conclude that $f(k)f(n) = f(f(k)f(n))$. Applying $R(f(n), k)$ we deduce that $f(n)$ is coprime to $f(k)$, as required. □

Claim. If p is prime then $f(p)$ is a power of p.

Proof. Suppose otherwise. We know that $p \mid f(p)$; let $q \neq p$ be another prime with $q \mid f(p)$.

If, for some positive integer N, we have $p \nmid f(N)$, then $f(p)$ is coprime to $f(N)$, so $q \nmid f(N)$, so $q \nmid N$; thus, if $q \mid N$, then $p \mid f(N)$ (and in particular, $p \mid f(q)$, by taking $N = q$).

Similarly, if $q \nmid f(N)$ then $f(q)$ is coprime to $f(N)$; as $p \mid f(q)$, this means $p \nmid f(N)$, so $p \nmid N$. So if $p \mid N$, then $q \mid f(N)$.

Together with $\operatorname{rad}(n) \mid f(n)$, this means that for any n not coprime to pq, we have $pq \mid f(n)$.

Let $m = \min\{\nu_p(f(x)) \mid x \text{ is not coprime to } pq\}$, and let X be a positive integer not coprime to pq such that $\nu_p(f(X)) = m$. The argument above shows $m \geqslant 1$. We can write $f(X) = p^m q^y X'$, where $y \geqslant 1$, $p \nmid X'$ and $q \nmid X'$. Since $f(f(X)) = f(X)$ we have $f(p^m q^y X') = p^m q^y X'$. Applying $Q(p^m, q^y X')$ gives $(p^m q^y X')^2 = f(p^m) f(q^y X') f(p^m f(q^y X'))$. The RHS is divisible by p^{3m} but the LHS is only divisible by p^{2m}, yielding a contradiction. □

Claim. For any integer n, $\operatorname{rad}(f(n)) = \operatorname{rad}(n)$.

Proof. We already have that $\operatorname{rad}(n) \mid f(n)$, so it remains only to show that no other primes divide $f(n)$. If p is prime and $p \nmid n$, the previous Claim shows that n is coprime to $f(p)$, and thus $f(n)$ is coprime to $f(p)$; that is, $p \nmid f(n)$. So exactly the same primes divide $f(n)$ as divide n. □

It remains only to exhibit functions that show all values of $f(n)$ with $\operatorname{rad}(f(n)) = \operatorname{rad}(n)$ are possible. Given $e(p) \geqslant 1$ for each prime p, take

$$f(n) = \prod_{p \mid n} p^{e(p)}$$

and we verify by examining exponents of each prime that this satisfies the conditions of the problem.

Comment. A quicker but less straightforward proof that $f(1) = 1$ is to let $f(n) = M$ be the least value that f takes; then $P(1, n)$ gives $M^2 = f(n)^2 = f(1) f(f(n))^2 \geqslant M^3$ so $M = 1$ and $f(1) = 1$.

Solution 2. As in Solution 1, we see that there are indeed functions f satisfying the given condition and producing all the given values of $f(n)$, and we follow Solution 1 to show the following facts:

- $f(1) = 1$.
- $f(m) = f(m^2)$ for all m.
- $f(n) = f(f(n))$ for all n.
- $f(mn)^2 = f(m) f(n) f(m f(n))$ if and only if m and n are coprime; refer to this as $Q(m, n)$.

Taking $Q(m, n)$ together with $Q(n, m)$ gives that $f(m f(n)) = f(n f(m))$ if m and n are coprime.

Suppose now that m is coprime to both n and $f(n)$. We have $f(mn)^2 = f(m) f(n) f(m f(n))$ and squaring both sides gives

$$\begin{aligned} f(mn)^4 &= f(m)^2 f(n)^2 f(m f(n))^2 \\ &= f(m)^2 f(n)^2 f(m) f(f(n)) f(m f(f(n))) \\ &= f(m)^3 f(n)^3 f(m f(n)). \end{aligned}$$

Thus $f(m f(n)) = f(m) f(n)$, so $f(mn)^2 = f(m)^2 f(n)^2$, so $f(mn) = f(m) f(n) = f(m f(n)) = f(n f(m))$.

If m is coprime to both n and $f(n)$ but however n is not coprime to $f(m)$, we have

$$\begin{aligned} f(nf(m))^2 &\neq f(n)f(f(m))f(nf(f(m))) \\ &= f(n)f(m)f(nf(m)) \\ &= f(nf(m))^2, \end{aligned}$$

a contradiction. Thus, given that m and n are coprime, we know that m is coprime to $f(n)$ if and only if n is coprime to $f(m)$. In particular, if p and q are different primes, then $p \mid f(q)$ if and only if $q \mid f(p)$, and likewise, for any positive integer k, $p \mid f(q^k)$ if and only if $q \mid f(p)$. More generally, if $p \nmid n$, then $p \mid f(n)$ if and only if n is not coprime to $f(p)$.

Now form a graph whose vertices are the primes, and where there is an edge between primes $p \neq q$ if and only if $p \mid f(q)$ (and so $q \mid f(p)$); every vertex has finite degree. For any integer n, the primes dividing $f(n)$ are all the primes that are neighbours of any prime $q \mid n$, together possibly with some further primes $p \mid n$.

If p and q are different primes, we have $f(pf(q)) = f(qf(p))$. The LHS is divisible by all primes that (in the graph) are neighbours of p or neighbours of neighbours of q, and possibly also by p and by some primes that are neighbours of q, and a corresponding statement with p and q swapped applies to the RHS. Thus any prime that is a neighbour of a neighbour of q must be one of: p, q, distance 1 from q, or distance 1 or 2 from p. For any prime r that is distance 2 from q, there are only finitely many primes p that it is distance 2 or less from, so by choosing a suitable prime p (depending on q) we conclude that every prime that is a neighbour of a neighbour of q is actually q itself or a neighbour of q.

So the connected components of the graph are (finite) complete graphs. If m is divisible only by primes in one component, and n is divisible only by primes in another component, then $f(mn) = f(m)f(n)$. If n is divisible by more than one prime from a component, considering the expression for $f(mn)^2$ as applied with successive prime power divisors of n shows that $f(n)$ is divisible by all the primes in that component. However, while $f(p^k)$ is divisible by all the primes in the component of p except possibly for p itself, we do not yet know that $p \mid f(p^k)$. We now consider cases for the order of a component.

For any prime p, we cannot have $f(p^k) = 1$, because $Q(p^k, p^k)$ gives

$$f(p^{2k})^2 \neq f(p^k)f(p^k)f(p^k f(p^k)),$$

and simplifying using $f(m^2) = f(m)$ results in $1 \neq 1$. So for a component of order 1, $f(p^k)$ is a positive power of p, so has the same set of prime factors as p, as required.

Now consider a component of order at least 2. Since $f(f(n)) = f(n)$, if the component has order at least 3, then for any $n \neq 1$ whose prime divisors are in that component, $f(n)$ is divisible by all the primes in that component. If the component has order 2, we saw above that this is true except possibly for $n = p^k$. However, if the primes in the component are p and q, and $f(p^k) = q^\ell$, then $f(q^\ell) = f(f(p^k)) = f(p^k) = q^\ell$, which contradicts $p \mid f(q^\ell)$. So for any component of order at least 2, and any $n \neq 1$ whose prime divisors are in that component, $f(n)$ is divisible by all the primes in that component.

In a component of order at least 2, let m be the product of all the primes in that component, and let t be maximal such that $m^t \mid f(n)$ for all $n \neq 1$ whose prime divisors are in that component; we have seen that $t \geqslant 1$. If m and n are coprime numbers greater than 1, all of whose prime divisors are in that component, then $Q(m, n)$ tells us that $m^{3t/2} \mid f(mn)$. For any $n' \neq 1$, all of whose prime divisors are in that component, $f(n')$ is divisible by all the primes in that component, so can be expressed as such a product, so $m^{3t/2} \mid f(f(n')) = f(n')$. But this means $t \geqslant 3t/2$, a contradiction, so all components have order 1, and we are done.

The activities of the Problem Selection Committee
were supported by the University of Cambridge

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